1–10 Sketch the vector field **F** by drawing a diagram like Figure 5 or Figure 9.

5. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

Note that

$$|F(x,y)| = \left|\frac{yi + xj}{\sqrt{x^2+y^2}}\right|$$

$$= \sqrt{\left(\frac{1}{\sqrt{x^2+y^2}}\right)^2 (y^2+x^2)}$$

$$= \sqrt{\frac{y^2+x^2}{x^2+y^2}}$$

$$= 1$$

A good strategy is to plot the vector field on lines (vertical or horizontal lines or slant) and well known curves (eg. $y = x^2$, $x = y^2$).

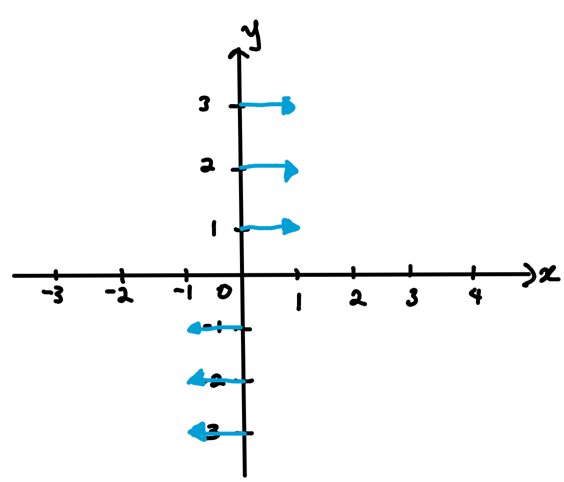
On $x = 0$,

$$F(x,y) = \frac{yi + 0j}{\sqrt{0^2+y^2}}$$

$$= \frac{yi}{\sqrt{y^2}}$$

$$= \frac{yi}{|y|}$$

$$= \frac{y}{|y|}\langle 1, 0\rangle$$

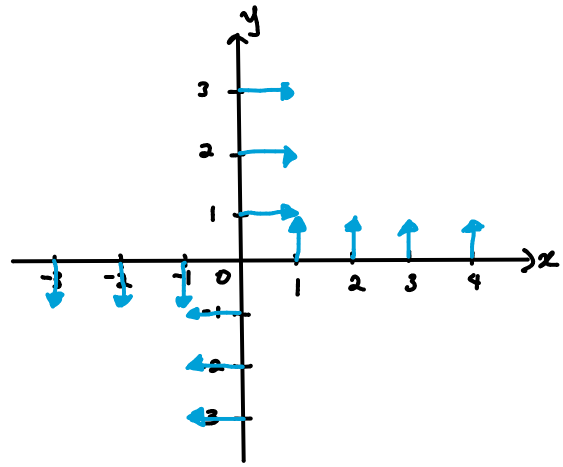


On $y = 0$,

$$F(x,y) = \frac{0i + xj}{\sqrt{x^2 + 0^2}}$$

$$= \frac{xj}{|x|}$$

$$= \frac{x}{|x|}\langle 0, 1\rangle$$



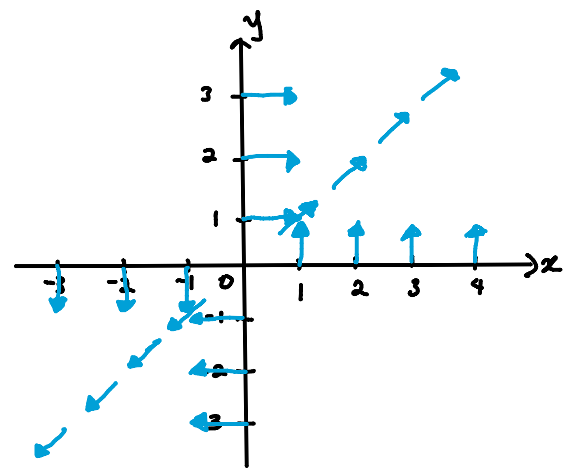
On $y = x$,

$$F(x,y) = \frac{xi + xj}{\sqrt{x^2 + x^2}}$$

$$= \frac{x(i+j)}{\sqrt{2x^2}}$$

$$= \frac{x}{|x|\sqrt{2}}(i+j)$$

$$= \frac{x}{|x|\sqrt{2}}\langle 1, 1\rangle$$



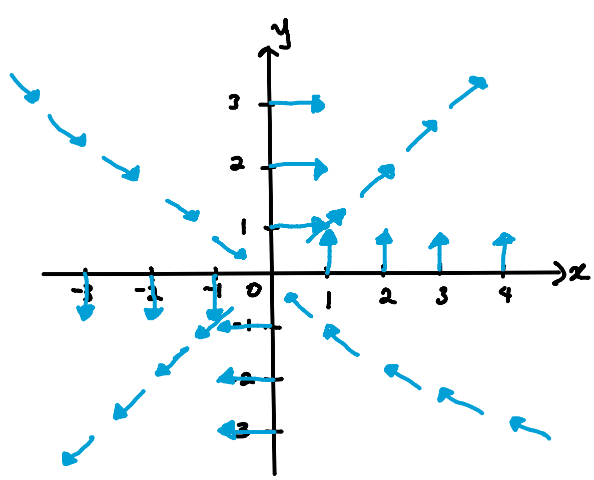
On $y = -x$,

$$F(x,y) = \frac{-xi + xj}{\sqrt{x^2 + x^2}}$$

$$= \frac{x(-i+j)}{\sqrt{2x^2}}$$

$$= \frac{x}{|x|\sqrt{2}}(-i+j)$$

$$= \frac{x}{|x|\sqrt{2}}\langle -1, 1\rangle$$



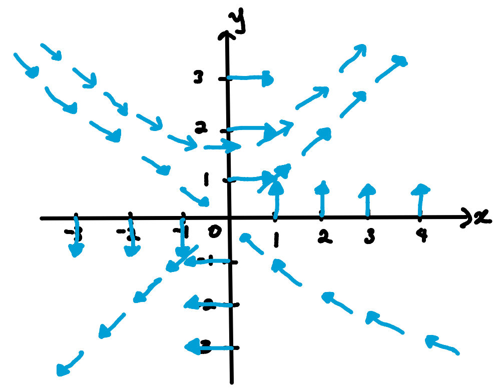
$$= \frac{x}{|x|\sqrt{2}} \langle -1, 1 \rangle$$

On $y = x^2$

$$F(x,y) = \frac{x^2 i + x j}{\sqrt{x^2 + x^4}}$$

$$= \frac{x(x i + j)}{|x|\sqrt{1 + x^2}}$$

$$= \frac{x}{|x|\sqrt{1+x^2}} \langle x, 1 \rangle$$

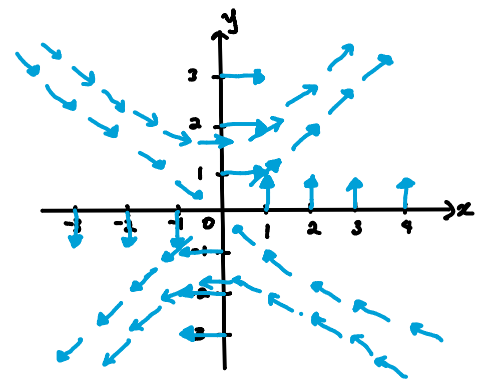


$a > 0$

More generally,

$y = ax^2$

$$\Rightarrow F(x,y) = \frac{1}{\sqrt{x^2+x^2}} \langle ax^2, x \rangle$$



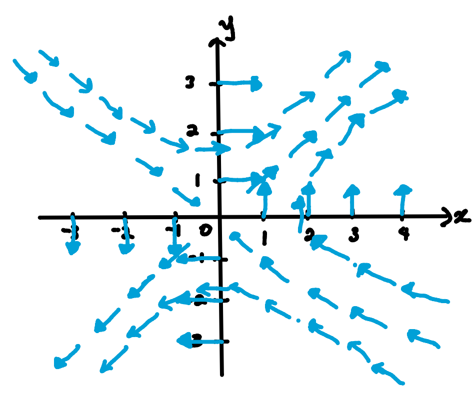
$a < 0$

Similarly,

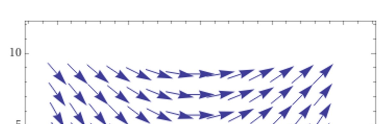
On $x = ay^2$,

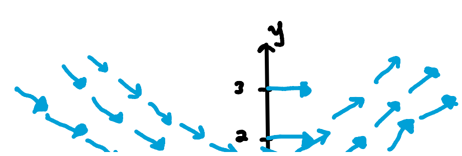
$$F(x,y) = \frac{y i + a y^2 j}{\sqrt{y^4 + y^2}}$$

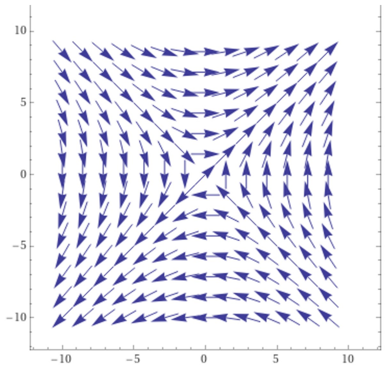
$$= \frac{1}{\sqrt{y^4+y^2}} \langle y, ay^2 \rangle$$



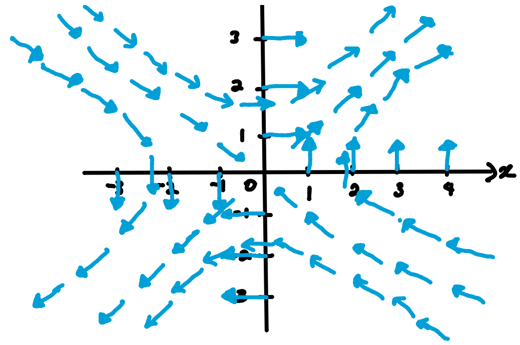
$a > 0$







Plot from WolframAlpha



$a < 0$

33. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position (2, 1) at time $t = 3$, estimate its location at time $t = 3.01$.

At time $t = 3$, the position, $r(t) = \langle x(t), y(t) \rangle$ of the particle is $\langle 2, 1 \rangle$.
So velocity of the particle at $t = 3$ is

$$v(3) = \langle 2^2, 2 + 1^2 \rangle$$
$$= \langle 4, 3 \rangle.$$

$$r(t) = \langle x(t), y(t) \rangle$$
$$dr(t) = \langle x'(t), y'(t) \rangle dt$$

But
$$r(t) = \langle x(t), y(t) \rangle$$
$$\Rightarrow \frac{dr(t)}{dt} = \langle x'(t), y'(t) \rangle$$
$$\Rightarrow dr(t) = \langle x'(t), y'(t) \rangle dt$$
$$= v(t)\, dt$$

Thus, change in position
$$dr(3) = v(3)\,(0.01)$$

$$= \langle 4, 3 \rangle \, 0.01$$

$$= \langle 0.04, 0.03 \rangle$$

Hence, at $t = 3.01$, position of the particle

$$r(3.01) = r(3) + \Delta r(3)$$

$$= \langle 2, 1 \rangle + \langle 0.04, 0.03 \rangle$$

$$= \langle 2.04, 2.03 \rangle.$$

Section 16.2: 2, 13, 21, 33, 38

1–16 Evaluate the line integral, where C is the given curve.

2. $\int_C (x/y)\, ds, \quad C\colon x = t^3,\ y = t^4,\ 1 \leqslant t \leqslant 2$

By definition,

$$\int_C \frac{x}{y}\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x(t) = t^3 \Rightarrow \frac{dx}{dt} = 3t^2$

$y(t) = t^4 \Rightarrow \frac{dy}{dt} = 4t^3$

$$= \int_1^2 \frac{t^3}{t^4} \sqrt{(3t^2)^2 + (4t^3)^2}\, dt$$

$$= \int_1^2 \frac{1}{t} \sqrt{9t^4 + 16t^6}\, dt$$

$$= \int_1^2 \frac{1}{t} \sqrt{t^4(9 + 16t^2)}\, dt$$

$$= \int_1^2 \frac{1}{t} \left(\sqrt{t^4}\right) \sqrt{9 + 16t^2}\, dt$$

$$= \int_1^2 \frac{1}{t} (t^2) \sqrt{9 + 16t^2}\, dt$$

$$= \int_1^2 t \sqrt{9 + 16t^2}\, dt$$

$u = 9 + 16t^2$

$\Rightarrow du = 32t\,dt$

$\Rightarrow dt = \frac{du}{32t}$

Also,

$t = 1 \Rightarrow u = 9 + 16 = 25$

$t = 2 \Rightarrow u = 9 + 16(2^2) = 73$

$$= \int_{25}^{73} t \cdot u^{1/2} \cdot \frac{du}{32t}$$

$$= \frac{1}{32} \int_{25}^{73} u^{1/2}\, du$$

25

$$= \frac{1}{32}\left(\frac{2}{3}u^{3/2}\right)\Big|_{25}^{73}$$

$$= \frac{1}{48}\left(73^{3/2} - 25^{3/2}\right)$$

$$= \frac{1}{48}\left(73\cdot 73^{1/2} - 5^3\right)$$

$$= \frac{1}{48}\left(73\sqrt{73} - 125\right).$$

13. $\int_C xye^{yz}\,dy, \quad C\colon x = t,\ y = t^2,\ z = t^3,\ 0 \leqslant t \leqslant 1$

By definition,

$$\int_C xye^{yz}\,dy = \int_a^b f\left(x(t), y(t), z(t)\right) y'(t)\,dt$$

$$= \int_0^1 t(t^2)e^{t^2\cdot t^3}\cdot(2t)\,dt$$

$$= 2\int_0^1 t^4 e^{t^5}\,dt$$

$u = t^5$

$\Rightarrow du = 5t^4\,dt$

$\Rightarrow dt = \dfrac{du}{5t^4}$

Also,

$t = 0 \Rightarrow u = 0$

$t = 1 \Rightarrow u = 1$

$$= 2\int_0^1 t^4 e^u\cdot\frac{du}{5t^4}$$

$$= \frac{2}{5}\int_0^1 e^u\,du$$

$$= \frac{2}{5}e^u\Big|_{u=0}^{u=1}$$

$$= \frac{2}{5}\left(e^1 - e^0\right)$$

$$= \frac{2}{5}\left(e^1 - 1\right).$$

19–22 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

21. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k}$,
$\mathbf{r}(t) = t^3\,\mathbf{i} - t^2\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

By definition,

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$$

$$= \int_a^b F(x(t), y(t), z(t)) \cdot r'(t)\,dt$$

But

$$r(t) = t^3 i - t^2 j + tk$$

$$= \langle t^3, -t^2, t \rangle$$

$$\Rightarrow r'(t) = \langle 3t^2, -2t, 1 \rangle \text{ and } x(t) = t^3,\ y(t) = -t^2,\ z(t) = t.$$

$$\Rightarrow F(x, y, z) = \langle \sin t^3, \cos(-t^2), t^3 \cdot t \rangle$$

$$= \langle \sin t^3, \cos t^2, t^4 \rangle.$$

Hence,

$$\int F \cdot dr = \int^b \ldots\, r'(t)\,dt$$

$$\int_C F\cdot dr = \int_a^b F(x(t),y(t),z(t))\cdot r'(t)\,dt$$

$$= \int_0^1 \langle \sin t^3,\ \cos t^2,\ t^4\rangle \cdot \langle 3t^2, -2t, 1\rangle\,dt$$

$$= \int_0^1 \left[3t^2\sin t^3 - 2t\cos t^2 + t^4\right]dt$$

$$= 3\int_0^1 t^2 \sin t^3\,dt - 2\int_0^1 t\cos t^2\,dt + \int_0^1 t^4\,dt$$

$u = t^3 \Rightarrow du = 3t^2dt$
$\Rightarrow dt = \frac{du}{3t^2}$

$$= 3\int_0^1 t^2 \sin u\cdot \frac{du}{3t^2} - 2\int_0^1 t\cos u\cdot\frac{du}{2t} + \frac{1}{5}t^5\Big|_{t=0}^{t=1}$$

$u = t^2 \Rightarrow du = 2t\,dt$
$\Rightarrow dt = \frac{du}{2t}$

In each case,
$t=0 \Rightarrow u=0,$
$t=1 \Rightarrow u=1$

$$= \int_0^1 \sin u\,du - \int_0^1 \cos u\,du + \frac{1}{5}(1^5 - 0^5)$$

$$= -\cos u\Big|_{u=0}^{u=1} - \sin u\Big|_{u=0}^{u=1} + \frac{1}{5}$$

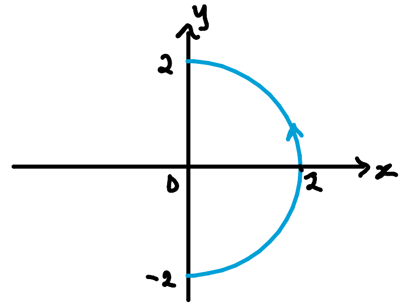
$$= -\cos 1 + \cos 0 - \sin 1 + \sin 0 + \frac{1}{5}$$

$$= -\cos 1 + 1 - \sin 1 + \frac{1}{5}$$

$$= 1 + \frac{1}{5} - \cos 1 - \sin 1$$

$$= \frac{6}{5} - \cos 1 - \sin 1.$$

33. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \geqslant 0$. If the linear density is a constant k, find the mass and center of mass of the wire.



Using the parametrization

$x = 2\cos t$ and $y = 2\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ (Think polar!).

By definition, the mass

$$m = \int_C \rho(x,y)\,ds$$

$$= \int_{-\pi/2}^{\pi/2} k \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{-\pi/2}^{\pi/2} k \sqrt{(2\sin t)^2 + (-2\cos t)^2}\,dt$$

$$= \int_{-\pi/2}^{\pi/2} k \sqrt{4(\sin^2 t + \cos^2 t)}\,dt$$

$$= \int_{-\pi/2}^{\pi/2} k \underbrace{(2)\,dt}_{ds}$$

$$= 2k \int_{-\pi/2}^{\pi/2} dt$$

$$= 2k\,(t)\Big|_{t=-\pi/2}^{\pi/2}$$

$$= 2k\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= 2\pi k.$$

The center of mass is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{m} \int_C x\rho(x,y)\,ds$$

$$= \frac{1}{2\pi k} \int_{-\pi/2}^{\pi/2} x k \cdot \underbrace{2\,dt}_{ds}$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 2\cos t\,dt$$

$$= \frac{2}{\pi} \left(\sin t\right) \Big|_{t=-\pi/2}^{\pi/2}$$

$$= \frac{2}{\pi} \left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{2}{\pi} (1+1)$$

$$= \frac{4}{\pi}.$$

and

$$\bar{y} = \frac{1}{m} \int_C y\rho(x,y)\,ds$$

$$= \frac{1}{2\pi k} \int_{-\pi/2}^{\pi/2} y\, k \cdot \underbrace{2\,dt}_{ds}$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 2\sin t\,dt$$

$$= \frac{2}{\pi}\left(-\cos t\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= -\frac{2}{\pi}\left(\cos\left(\tfrac{\pi}{2}\right) - \cos\left(-\tfrac{\pi}{2}\right)\right)$$

$$= -\frac{2}{\pi}\left(0 - 0\right)$$

$$= 0$$

Hence, the center of mass

$$(\bar{x}, \bar{y}) = \left(\frac{4}{\pi}, 0\right).$$

38. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C, its **moments of inertia** about the x-, y-, and z-axes are defined as

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds$$

Find the moments of inertia for the wire in Exercise 35.

35. (b) Find the center of mass of a wire in the shape of the helix $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \leqslant t \leqslant 2\pi$, if the density is a constant k.

In 3D,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

In 3D,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \sqrt{(2\cos t)^2 + (-2\sin t)^2 + 3^2}\, dt$$

$$= \sqrt{4(\cos^2 t + \sin^2 t) + 9}\quad dt$$

$$= \sqrt{4+9}\quad dt$$

$$= \sqrt{13}\, dt$$

Thus, by definition, the moments of inertia are

$$I_x = \int_C (y^2 + z^2)\, \rho(x,y,z)\, ds$$

$$= \int_0^{2\pi} \left[(2\cos t)^2 + (3t)^2\right] k \cdot \sqrt{13}\, dt$$

$$= k\sqrt{13} \int_0^{2\pi} \left(4\cos^2 t + 9t^2\right) dt$$

$\cos^2 t = \frac{1}{2}(1+\cos(2t))$

$$= k\sqrt{13} \int_0^{2\pi} \left[4 \cdot \frac{1}{2}(1+\cos(2t)) + 9t^2\right] dt$$

$$= k\sqrt{13} \int_0^{2\pi} \left(2 + 2\cos 2t + 9t^2\right) dt$$

$$= k\sqrt{13} \left[2t + \sin 2t + 3t^3\right]_{t=0}^{t=2\pi}$$

$$= k\sqrt{13} \left[2(2\pi) - 2(0) + \sin 4\pi - \sin 0 + 3(2\pi)^3 - 3(0^3)\right]$$

$$= k\sqrt{13}\left(4\pi + 24\pi^3\right)$$

$$= 4\pi k\sqrt{13}\left(1 + 6\pi^2\right).$$

$$I_y = \int_C \left(x^2+z^2\right)\rho(x,y,z)\,ds$$

$$= \int_0^{2\pi} \left[(2\sin t)^2 + (3t)^2\right] k \cdot \sqrt{13}\,dt$$

$$= k\sqrt{13}\int_0^{2\pi} \left[4\sin^2 t + 9t^2\right] dt$$

$$= k\sqrt{13}\int_0^{2\pi} \left[4\left(1-\cos^2 t\right) + 9t^2\right] dt$$

$$= k\sqrt{13}\int_0^{2\pi} \left[4 - 4\cdot\tfrac{1}{2}\left(1+\cos(2t)\right) + 9t^2\right] dt$$

$$= k\sqrt{13}\int_0^{2\pi} \left[4 - 2 - 2\cos 2t + 9t^2\right] dt$$

$$= k\sqrt{13}\int_0^{2\pi} \left(2 - 2\cos 2t + 9t^2\right) dt$$

$$= k\sqrt{13}\left[2t - 2\sin 2t + 3t^3\right]_{t=0}^{2\pi}$$

$$= k\sqrt{13}\left[2(2\pi) - 2(0) - 2\sin 4\pi + 2\sin 0 + 3(2\pi)^3 - 3(0)^3\right]$$

$$= k\sqrt{13}\left[4\pi + 24\pi^3\right]$$

$$= 4\pi k\sqrt{13}\left(1 + 6\pi^2\right).$$

$$I_z = \int_C (x^2+y^2)\,\rho(x,y,z)\,ds$$

$$= \int_0^{2\pi}\left[(2\sin t)^2 + (2\cos t)^2\right] k\cdot\sqrt{13}\,dt$$

$$= \int_0^{2\pi}\left[4\,(\sin^2 t + \cos^2 t)\right] k\sqrt{13}\,dt$$

$$= 4k\sqrt{13}\int_0^{2\pi} dt$$

$$= 4k\sqrt{13}\; t\Big|_{t=0}^{2\pi}$$

$$= 8\pi\sqrt{13}\,k.$$