1–6 Find the length of the curve.

2. $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle, \quad 0 \leqslant t \leqslant 1$

$$r'(t) = \langle 2, 2t, t^2 \rangle.$$

$$|r'(t)| = \sqrt{2^2 + (2t)^2 + (t^2)^2} = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2}$$

$$= |2 + t^2|$$

$$= 2 + t^2.$$

Thus, length of the curve

$$L = \int_0^1 |r'(t)|\,dt = \int_0^1 (2 + t^2)\,dt$$

$$= 2t + \frac{1}{3}t^3 \Big|_{t=0}^{t=1}$$

$$= \left(2(1) + \frac{1}{3}(1^3)\right) - \left(2(0) + \frac{1}{3}(0^3)\right)$$

$$= 2 + \frac{1}{3}$$

$$= \frac{7}{3}$$

4. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln \cos t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi/4$

$$r'(t) = -\sin t\,i + \cos t\,k - \frac{\sin t}{\cos t}\,k = -\sin t\,i + \cos t\,k - \tan t\,k$$

$$\Rightarrow |r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$$

$$= \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$$

$$= \sqrt{1 + \tan^2 t}$$

$$= \sqrt{\sec^2 t}$$

$$= |\sec t|$$

$$= \sec t \quad \text{since } \sec t \geq 0 \text{ for } 0 \leq t \leq \frac{\pi}{4}$$

So the arclength

$$L = \int_0^{\pi/4} \sec t \, dt$$

$$= \ln|\tan t + \sec t| \Big|_0^{\pi/4}$$

$$= \ln\left|\tan \tfrac{\pi}{4} + \sec \tfrac{\pi}{4}\right| - \ln|\tan 0 + \sec 0|$$

$$= \ln|1 + \sqrt{2}| - \ln|0 + 1|$$

$$= \ln(1 + \sqrt{2})$$

17–20

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

(b) Use Formula 9 to find the curvature.

19. $\mathbf{r}(t) = \left\langle \sqrt{2}\, t, e^t, e^{-t} \right\rangle$

$$r'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle.$$

$$\Rightarrow |r'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}$$

$$\Rightarrow |r'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}$$

$$= \sqrt{2 + e^{2t} + e^{-2t}}$$

$$= \sqrt{(e^t + e^{-t})^2}$$

$$= |e^t + e^{-t}|$$

$$= e^t + e^{-t} \quad \text{since } e^t, e^{-t} > 0 \text{ for all } t.$$

So, the unit tangent vector

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$= \frac{1}{e^t + e^{-t}} \langle \sqrt{2}, e^t, -e^{-t} \rangle$$

$$= \frac{1}{e^t + e^{-t}} \langle \sqrt{2}, e^t, -\frac{1}{e^t} \rangle \cdot \frac{e^t}{e^t}$$

$$= \frac{1}{e^{2t} + 1} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle.$$

Furthermore,

$$T'(t) = \frac{1}{e^{2t}+1} \langle \sqrt{2}e^t, 2e^{2t}, 0 \rangle - \frac{2e^{2t}}{(e^{2t}+1)^2} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle.$$

$$= \frac{1}{(e^{2t}+1)^2} \left((e^{2t}+1) \langle \sqrt{2}e^t, 2e^{2t}, 0 \rangle - 2e^{2t} \langle \sqrt{2}e^t, e^{2t}, -1 \rangle \right)$$

$$= \frac{1}{(e^{2t}+1)^2} \left(\langle \sqrt{2}e^{3t} + \sqrt{2}e^t, 2e^{4t} + 2e^{2t}, 0 \rangle - \langle 2\sqrt{2}e^{3t}, 2e^{4t}, -2e^{2t} \rangle \right)$$

$$= \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^{t} - \sqrt{2}e^{3t},\ 2e^{2t},\ 2e^{2t} \right\rangle$$

$$= \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^{t}(1-e^{2t}),\ 2e^{2t},\ 2e^{2t} \right\rangle.$$

and

$$|T'(t)| = \sqrt{\left(\frac{\sqrt{2}e^{t}(1-e^{2t})}{(e^{2t}+1)^2}\right)^2 + \left(\frac{2e^{2t}}{(e^{2t}+1)^2}\right)^2 + \left(\frac{2e^{2t}}{(e^{2t}+1)^2}\right)^2}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1-e^{2t})^2 + 4e^{4t} + 4e^{4t}}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1-2e^{2t}+e^{4t}) + 8e^{4t}}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t} - 4e^{4t} + 2e^{6t} + 8e^{4t}}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t} + 4e^{4t} + 2e^{6t}}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1+2e^{2t}+e^{4t})}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1+e^{2t})^2}$$

$$= \frac{\sqrt{2}\, e^{t}(1+e^{2t})}{(e^{2t}+1)^2}$$

$$= \frac{\sqrt{2}\, e^{t}}{e^{2t}+1}$$

Thus, the unit normal vector

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$$= \frac{1}{(e^{2t}+1)^2} \left\langle \sqrt{2}e^{t}(1-e^{2t}),\ 2e^{2t},\ 2e^{2t} \right\rangle \times \frac{e^{2t}+1}{\sqrt{2}e^{t}}$$

$$= \frac{1}{\sqrt{2}e^{t}(e^{2t}+1)} \left\langle \sqrt{2}e^{t}(1-e^{2t}),\ 2e^{2t},\ 2e^{2t} \right\rangle.$$

Also, by formula 9 in the book, the curvature

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|}$$

$$= \frac{\sqrt{2}\, e^{t}}{e^{2t}+1} \cdot \frac{1}{e^{t}+e^{-t}}$$

$$= \frac{\sqrt{2}\, e^{t}}{e^{3t}+e^{t}+e^{t}+e^{-t}}$$

$$= \frac{\sqrt{2}\, e^{t}}{e^{3t}+2e^{t}+e^{-t}}$$

30–31 At what point does the curve have maximum curvature? What happens to the curvature as $x \to \infty$?

30. $y = \ln x$

$y = \ln x \Rightarrow y'(x) = \frac{1}{x}$ and $y''(x) = -\frac{1}{x^2}$.

So by formula (11) in the book, the curvature

$$k(x) = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}}$$

$$= \frac{\left|-\frac{1}{x^2}\right|}{\left[1 + \left(\frac{1}{x}\right)^2\right]^{3/2}}$$

$$= \frac{1/x^2}{\left(1 + 1/x^2\right)^{3/2}}$$

$$= \frac{1}{x^2\left(\frac{x^2+1}{x^2}\right)^{3/2}}$$

$$= \frac{1}{x^2 \frac{(x^2+1)^{3/2}}{(x^2)^{3/2}}}$$

$$= \frac{1}{\frac{(x^2+1)^{3/2}}{x}}$$

$$= \frac{x}{(x^2+1)^{3/2}}$$

To find point of maximum curvature, we need to find the critical numbers.

To find point of maximum curvature, we need to find the critical numbers.

$$k'(x) = \frac{(x^2+1)^{3/2} - \frac{3}{2}(2x)(x^2+1)^{1/2}(x)}{(x^2+1)^3}$$

$$= \frac{(x^2+1)^{1/2}\left[(x^2+1) - 3x^2\right]}{(x^2+1)^3}$$

$$= \frac{(x^2+1)^{1/2}(1-2x^2)}{(x^2+1)^3}$$

So $k(x) = 0 \Rightarrow 1-2x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow x = \frac{1}{\sqrt{2}}$ since $-\frac{1}{\sqrt{2}}$ is not in the domain of y.

Also, $k'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $k'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$

$\Rightarrow k(x)$ attains maximum value at $x = \frac{1}{\sqrt{2}}$.

(Alternatively: Compute $\kappa''(x)$ to see that $\kappa''(\frac{1}{\sqrt{2}}) < 0$)

Moreover,

$$\lim_{x\to\infty} k(x) = \lim_{x\to\infty} \frac{x}{(x^2+1)^{3/2}}$$

$$= \lim_{x\to\infty} \frac{x}{x^3\left(1+\frac{1}{x^2}\right)^{3/2}}$$

$$= \lim_{x\to\infty} \frac{1/x^2}{\left(1+\frac{1}{x^2}\right)^{3/2}}$$

$$= \frac{\lim\limits_{x\to\infty} 1/x^2}{\left(1+\lim\limits_{x\to\infty}\frac{1}{x^2}\right)^{3/2}}$$

$$= \frac{0}{(1+0)^{3/2}} = 0.$$