1–8 Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = \underbrace{xy^2z^2}_{P}\,\mathbf{i} + \underbrace{x^2yz^2}_{Q}\,\mathbf{j} + \underbrace{x^2y^2z}_{R}\,\mathbf{k}$

(a) The curl of F is defined as

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2y^2z) - \frac{\partial}{\partial z}(x^2yz^2)\right)i - \left(\frac{\partial}{\partial x}(x^2y^2z) - \frac{\partial}{\partial z}(xy^2z^2)\right)j + \left(\frac{\partial}{\partial x}(x^2yz^2) - \frac{\partial}{\partial y}(xy^2z^2)\right)k$$

$$= (2x^2yz - 2x^2yz)i - (2xy^2z - 2xy^2z)j + (2xyz^2 - 2xyz^2)k$$

$$= 0i - (0)j + 0k$$

$$= 0.$$

(b) The divergence of F

$$\operatorname{div} F = \nabla \cdot F$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (P, Q, R)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \frac{\partial}{\partial x}(xy^2z^2) + \frac{\partial}{\partial y}(x^2yz^2) + \frac{\partial}{\partial z}(x^2y^2z)$$

$$= y^2z^2 + x^2z^2 + x^2y^2.$$

13–18 Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

16. $\mathbf{F}(x, y, z) = \mathbf{i} + \sin z\,\mathbf{j} + y\cos z\,\mathbf{k}$

Since the domain of F is $\mathbb{R}^3$ and the components of F have continuous partial derivatives, it suffices to show curl F = 0.

But

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & \sin z & y\cos z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(y\cos z) - \frac{\partial}{\partial z}(\sin z)\right) - j\left(\frac{\partial}{\partial x}(y\cos z) - \frac{\partial}{\partial z}(1)\right) + k\left(\frac{\partial}{\partial x}(\sin z) - \frac{\partial}{\partial y}(1)\right)$$

$$= i(\cos z - \cos z) - j(0-0) + k(0-0)$$

$$= 0.$$

Hence, F is conservative by Theorem 4.

Now, we look for f such that $\nabla f = F$.

$$\Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (1, \sin z, y\cos z)$$

$$\Rightarrow \frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = \sin z, \quad \frac{\partial f}{\partial z} = y\cos z$$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial x} = 1,}_{\mathrm{I}} \quad \underbrace{\frac{\partial f}{\partial y} = \sin z,}_{\mathrm{II}} \quad \underbrace{\frac{\partial f}{\partial z} = y\cos z}_{\mathrm{III}}$$

$$(\mathrm{I}) \Rightarrow f(x,y,z) = \int 1 \, \partial x$$
$$= x + g(y,z).$$

Differenting wrt y and comparing with (II),

$$f_y(x,y,z) = g_y(y,z) = \sin z$$
$$\Rightarrow g(y,z) = \int \sin z \, \partial y$$
$$= y\sin z + h(z)$$

Thus,

$$f(x,y,z) = x + y\sin z + h(z).$$

Finally, differentiating f and equating to (III),

$$f_z(x,y,z) = y\cos z + h'(z) = y\cos z$$
$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = c, \text{ a constant.}$$

Hence,

$$f(x,y,z) = x + y\sin z + c$$

is the required f.

23–29 Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\,\mathbf{F}$, $\mathbf{F}\cdot\mathbf{G}$, and $\mathbf{F}\times\mathbf{G}$ are defined by

$$(f\,\mathbf{F})(x, y, z) = f(x, y, z)\,\mathbf{F}(x, y, z)$$

$$(\mathbf{F}\cdot\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\cdot\mathbf{G}(x, y, z)$$

$$(\mathbf{F}\times\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\times\mathbf{G}(x, y, z)$$

26. $\operatorname{curl}(f\,\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$

For convenience, let $F = Pi + Qj + RK$. Then $fF = fPi + fQj + fRK$.

Thus, by definition,

$$\operatorname{curl}(fF) = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fP & fQ & fR \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(fR) - \frac{\partial}{\partial z}(fQ)\right) - j\left(\frac{\partial}{\partial x}(fR) - \frac{\partial}{\partial z}(fP)\right) + K\left(\frac{\partial}{\partial x}(fQ) - \frac{\partial}{\partial y}(fP)\right)$$

Using Product Rule

$$= i\left(f\frac{\partial R}{\partial y} + R\frac{\partial f}{\partial y} - f\frac{\partial Q}{\partial z} - Q\frac{\partial f}{\partial z}\right) - j\left(f\frac{\partial R}{\partial x} + R\frac{\partial f}{\partial x} - f\frac{\partial P}{\partial z} - P\frac{\partial f}{\partial z}\right) + K\left(f\frac{\partial Q}{\partial x} + Q\frac{\partial f}{\partial x} - f\frac{\partial P}{\partial y} - P\frac{\partial f}{\partial y}\right)$$

$$= if\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + i\left(R\frac{\partial f}{\partial y} - Q\frac{\partial f}{\partial z}\right) - jf\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) - j\left(R\frac{\partial f}{\partial x} - P\frac{\partial f}{\partial z}\right) + Kf\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) + K\left(Q\frac{\partial f}{\partial x} - P\frac{\partial f}{\partial y}\right)$$

$$= \mathbf{i} f\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \mathbf{j} f\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \mathbf{k} f\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$+ \underbrace{\mathbf{i}\left(R\frac{\partial f}{\partial y} - Q\frac{\partial f}{\partial z}\right) - \mathbf{j}\left(R\frac{\partial f}{\partial x} - P\frac{\partial f}{\partial z}\right) + \mathbf{k}\left(Q\frac{\partial f}{\partial x} - P\frac{\partial f}{\partial y}\right)}_{\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ P & Q & R \end{vmatrix}}$$

$$= f\left[\mathbf{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \mathbf{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= f\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= f(\nabla \times F) + \nabla f \times F$$

$$= f \operatorname{curl} F + \nabla f \times F \quad \text{as required.}$$

30–32 Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

30. Verify each identity.

(a) $\nabla \cdot \mathbf{r} = 3$

By definition,

$$\nabla \cdot r = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x, y, z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1$$

$$= 3$$

(b) $\nabla \cdot (r\mathbf{r}) = 4r$

By definition,

$$\nabla \cdot (r\mathbf{r}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (xr, yr, zr)$$

$$= \frac{\partial}{\partial x}(xr) + \frac{\partial}{\partial y}(yr) + \frac{\partial}{\partial z}(zr)$$

$$= x\frac{\partial r}{\partial x} + r + y\frac{\partial r}{\partial y} + r + z\frac{\partial r}{\partial z} + r$$

$$= 3r + x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)$$

$$= 3r + \frac{x^2 + y^2 + z^2}{r}$$

$$= 3r + \frac{r^2}{r}$$

$$= 3r + r$$

$$= 4r \quad \text{as required.}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r_x = \frac{x}{r}$$

$$r_y = \frac{y}{r}$$

$$r_z = \frac{z}{r}$$

(c) $\nabla^2 r^3 = 12r$

By definition,

$$\nabla^2 r^3 = \nabla \cdot \nabla r^3$$

But

$$\frac{\partial}{\partial x} r^3 = 3r^2 \frac{\partial r}{\partial x} = 3r^2 \cdot \frac{x}{r} = 3rx$$

$$\frac{\partial}{\partial y} r^3 = 3r^2 \frac{\partial r}{\partial y} = 3r^2 \cdot \frac{y}{r} = 3ry$$

$$\frac{\partial}{\partial y} r^3 = 3r^3 \frac{\partial r}{\partial y} = 3r^2 \cdot \frac{y}{r} = 3ry$$

$$\frac{\partial r^3}{\partial z} = 3r^2 \frac{\partial r}{\partial z} = 3r^2 \cdot \frac{z}{r} = 3rz.$$

$$\Rightarrow \nabla r^3 = 3rxi + 3ryj + 3rzk$$

$$= 3r(xi + yj + zk)$$

$$= 3r\mathbf{r}$$

So

$$\nabla^2 r^3 = \nabla \cdot \nabla r^3$$

$$= \nabla \cdot (3r\mathbf{r})$$

$$= 3(\nabla \cdot (r\mathbf{r}))$$

$$= 3(4r) \qquad \text{by part } b$$

$$= 12r.$$