Section 14.6: 4, 7, 21, 32, 41, 49

4–6 Find the directional derivative of f at the given point in the direction indicated by the angle θ.

4. $f(x, y) = xy^3 - x^2, \quad (1, 2), \quad \theta = \pi/3$

A unit vector in the direction with angle $\theta = \pi/3$ is

$$u = \cos(\pi/3)\, i + \sin(\pi/3)\, j$$
$$= \frac{1}{2} i + \frac{\sqrt{3}}{2} j$$

Also, the gradient vector

$$\nabla f = \langle y^3 - 2x,\ 3xy^2 \rangle \Rightarrow \nabla f(1,2) = \langle 2^3 - 2(1),\ 3(1)(2^2) \rangle = \langle 6, 12 \rangle.$$

Hence, the directional derivative at $(1,2)$

$$\nabla f \cdot u = \langle 6, 12 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle$$
$$= 3 + 6\sqrt{3}$$

7–10

(a) Find the gradient of f.

(b) Evaluate the gradient at the point P.

(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = x/y, \quad P(2, 1), \quad \mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

(a) The gradient of f

$$\nabla f = \langle 1/y,\ -x/y^2 \rangle.$$

(b) The gradient evaluated at the point $P(2,1)$

$$\nabla f(2,1) = \langle 1/1,\ -2/1^2 \rangle = \langle 1, -2 \rangle.$$

(c) The required rate of change is the directional derivative of f along the unit vector u at the point $P(2,1)$.

$$\nabla f(2,1) \cdot \mathbf{u} = \langle 1, -2 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle$$
$$= \frac{3}{5} - \frac{8}{5}$$
$$= \frac{-5}{5}$$
$$= -1$$

21–26 Find the maximum rate of change of f at the given point and the direction in which it occurs.

21. $f(x, y) = 4y\sqrt{x}$, $(4, 1)$

The maximum rate of change of f occurs in the direction of the gradient vector ∇f and is the directional derivative of f along ∇f.

i.e;

$$\nabla f \cdot \mathbf{u} = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|.$$

Now,

$\nabla f = \langle \frac{2y}{\sqrt{x}}, 4\sqrt{x} \rangle \Rightarrow \nabla f(4,1) = \langle \frac{2(1)}{\sqrt{4}}, 4\sqrt{4} \rangle = \langle 1, 8 \rangle$ is a direction of maximum rate of change at the point $(4,1)$.

$\Rightarrow |\nabla f| = \sqrt{1+8^2} = \sqrt{65}$ is the maximum rate of change of f at $(4,1)$.

32. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where T is measured in °C and x, y, z in meters.

(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

(b) In which direction does the temperature increase fastest at P?

(c) Find the maximum rate of increase at P.

(a) Let $Q = (3,-3,3)$. Then

$$u = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle 3-2, -3+1, 3-2 \rangle}{|\langle 1,-2,1\rangle|} = \frac{1}{\sqrt{6}}\langle 1,-2,1\rangle$$

Also,

$$\nabla T = \langle -2x, -6y, -18z\rangle\, 200\, e^{-x^2-3y^2-9z^2}$$

$$= \langle -x, -3y, -9z\rangle\, 400 e^{-x^2-3y^2-9z^2}$$

$$\Rightarrow \nabla T(2,-1,2) = \langle -2, 3, -18\rangle\, 400 e^{-4-3-36}$$

$$= \langle -2, 3, -18\rangle\, 400 e^{-43}$$

Hence, the rate of change of the temperature at $P(2,-1,2)$ in the direction toward the point $(3,-3,3)$ is

$$\nabla f(2,-1,2)\cdot u = \langle -2, 3, -18\rangle\, 400 e^{-43} \cdot \langle 1,-2,1\rangle \frac{1}{\sqrt{6}}$$

$$= \frac{(-2-6-18)\, 400 e^{-43}}{\sqrt{6}}$$

$$= \frac{(-26)\, 400 e^{-43}}{\sqrt{6}}$$

$$= -10400 \quad {}^\circ C/m$$

$$= \frac{-10400}{\sqrt{6}\, e^{43}} \ {}^{0}C/m$$

(b) The temperature increases fastest in the direction of the gradient vector

$$\nabla T(2,-1,2) = \langle -2, 3, -18 \rangle 400 e^{-43}$$

$$= \langle -2, 3, -18 \rangle$$ since a scalar multiple of a vector does not change its direction.

(c) The maximum rate of change at P is

$$|\nabla T(2,-1,2)| = \sqrt{[(-2)^2 + 3^2 + (-18)^2](400 e^{-43})^2}$$

$$= 400 e^{-43} \sqrt{4+9+324}$$

$$= 400 e^{-43} \sqrt{337} \ {}^{0}C/m$$

41–46 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

41. $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10, \quad (3, 3, 5)$

(a) Let $F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2 - 10$. Then

$\nabla F = \langle 4(x-2), 2(y-1), 2(z-3) \rangle$ is a normal to the tangent plane.

$\Rightarrow \nabla F(3,3,5) = \langle 4(3-2), 2(3-1), 2(5-3) \rangle = \langle 4, 4, 4 \rangle$

Thus, equation of the tangent plane at $(3,3,5)$ is

$$4(x-3) + 4(y-3) + 4(z-5) = 0$$

$\Rightarrow 4x - 12 + 4y - 12 + 4z - 20 = 0$

$\Rightarrow 4x + 4y + 4z = 44.$

$\Rightarrow x + y + z = 11.$

(6) Equation of the normal line is

$$r = r_0 + tv$$

$$= \langle 3, 3, 5 \rangle + t \langle 4, 4, 4 \rangle$$

$$= \langle 3 + 4t, 3 + 4t, 5 + 4t \rangle$$

Or the parametric equations

$$x = 3 + 4t, \quad y = 3 + 4t, \quad z = 5 + 4t.$$

49. If $f(x, y) = xy$, find the gradient vector $\nabla f(3, 2)$ and use it to find the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

$\nabla f = \langle y, x \rangle \implies \nabla f(3, 2) = \langle 2, 3 \rangle.$

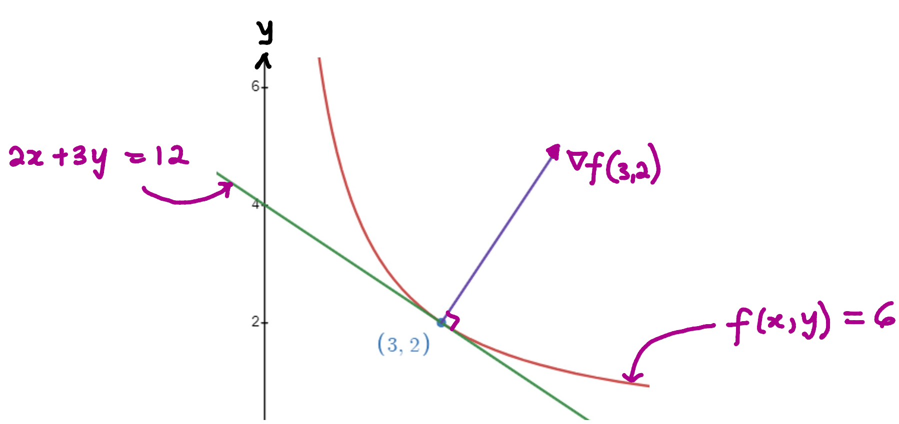
So

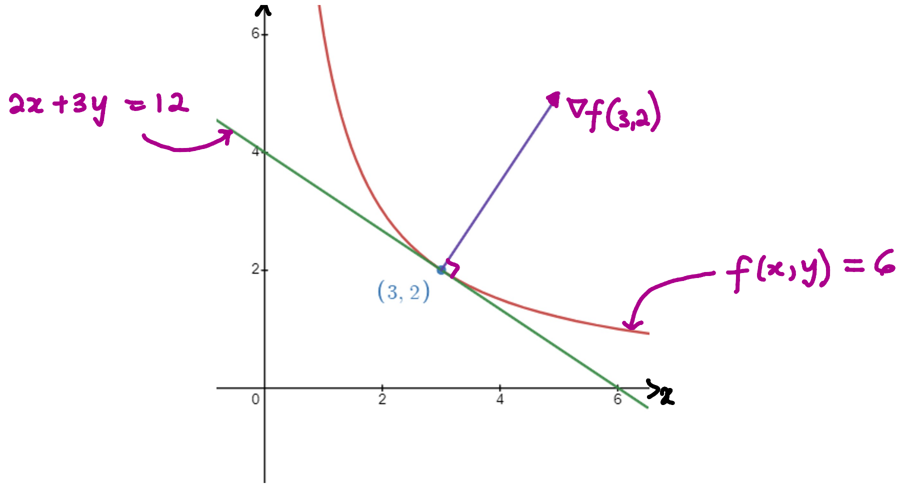
$$2(x - 3) + 3(y - 2) = 0$$

$$\Rightarrow 2x + 3y = 12$$

is the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$.

Also, the level curve $f(x, y) = 6 \Rightarrow xy = 6 \Rightarrow y = \frac{6}{x}$




2x + 3y = 12
∇f(3,2)
f(x,y) = 6
(3, 2)
x
0
2
4
6