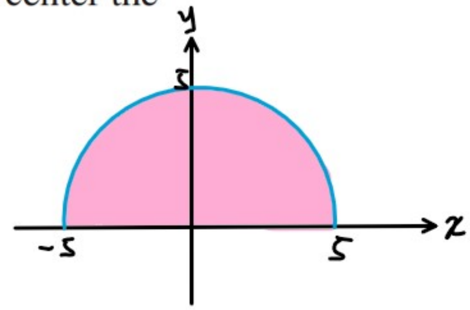
7–14 Evaluate the given integral by changing to polar coordinates.

7. $\iint_D x^2y\,dA$, where D is the top half of the disk with center the origin and radius 5



As shown, the half disk can be described as

$$D = \{(r,\theta) : 0 \le r \le 5,\ 0 \le \theta \le \pi\}$$

in polar coordinates.

Thus,

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\Rightarrow \iint_D x^2y\,dA = \int_0^{\pi}\int_0^{5} (r\cos\theta)^2 (r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\underbrace{\int_0^{5} r^4 \cos^2\theta \sin\theta\, dr}\,d\theta$$

$$= \int_0^{\pi} \left[\frac{r^5}{5}\cos^2\theta\sin\theta\right]_{r=0}^{5} d\theta$$

$$= \int_0^{\pi} \cos^2\theta\sin\theta\left[\frac{5^5}{5} - \frac{0^5}{5}\right] d\theta$$

$$= \int_0^{\pi} 625\cos^2\theta\sin\theta\,d\theta$$

Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$
$\Rightarrow d\theta = \dfrac{du}{-\sin\theta}$

$$= 625\int_0^{\pi} u^2 \sin\theta \cdot \frac{du}{-\sin\theta}$$

$$= -625\int_0^{\pi} u^2\,du$$

$$= -625\left[\frac{u^3}{3}\right]_{\theta=0}^{\theta=\pi}$$

$$= -625\left[\frac{u^3}{3}\right]_{\theta=0}^{\theta=\pi}$$

$$= -625\left[\frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= -\frac{625}{3}\left[\cos^3\pi - \cos^3 0\right]$$

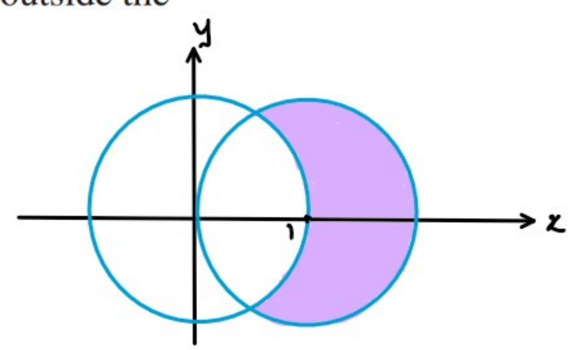
$$= -\frac{625}{3}(-1-1)$$

$$= \frac{625\times 2}{3}$$

$$= \frac{1250}{3}.$$

15–18 Use a double integral to find the area of the region.

17. The region inside the circle $(x-1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$



Using $x = r\cos\theta$ and $y = r\sin\theta$,

$(x-1)^2 + y^2 = 1$

$\Rightarrow x^2 + y^2 = 2x$

$\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 2r\cos\theta$ in polar coordinates

$\Rightarrow r^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} = 2r\cos\theta$

$\Rightarrow r^2 = 2r\cos\theta$

$\Rightarrow r = 2\cos\theta$

Also,

$$x^2+y^2=1 \Rightarrow (r\cos\theta)^2+(r\sin\theta)^2=1$$

$$\Rightarrow r^2(\cos^2\theta+\sin^2\theta)=1$$

$$\Rightarrow r^2=1$$

$$\Rightarrow r=1 \quad \text{Since } r\geq 0.$$

and the curves intersect when

$$2\cos\theta = r = 1$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$

$$\Rightarrow \theta = -\frac{\pi}{3}, \frac{\pi}{3}.$$

Thus, the region (shaded) is described in polar coordinates as

$$D=\left\{(r,\theta):\ 1\leq r\leq 2\cos\theta,\ -\frac{\pi}{3}\leq\theta\leq\frac{\pi}{3}\right\}$$

So that area of the region

$$\iint_D 1\,dA = \int_{-\pi/3}^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3}\left[\frac{r^2}{2}\right]_1^{2\cos\theta} d\theta$$

$$= \int_{-\pi/3}^{\pi/3}\left[\frac{(2\cos\theta)^2}{2} - \frac{1^2}{2}\right] d\theta$$

$$= \int_{-\pi/3}^{\pi/3}\left[2\underbrace{\cos^2\theta} - \frac{1}{2}\right] d\theta$$

$$= \int_{-\pi/3}^{\pi/3}\left[2\left(\frac{1}{2}(1+\cos(2\theta))\right) - \frac{1}{2}\right] d\theta$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\Rightarrow \cos(2\theta) = \cos^2 - \sin^2\theta$$

$$= \cos^2 - (1-\cos^2\theta)$$

$$= \int_{-\pi/3} \left[2\left(\tfrac{1}{2}(1+\cos(2\theta))\right) - \tfrac{1}{2}\right] d\theta$$

$\Rightarrow \cos(2\theta) = \ldots$

$= \cos^2 - (1 - \cos^2\theta)$

$= 2\cos^2\theta - 1$

$\Rightarrow \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$$= \int_{-\pi/3}^{\pi/3} \left[1 + \cos(2\theta) - \frac{1}{2}\right] d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left[\frac{1}{2} + \cos(2\theta)\right] d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{2}\sin(2\theta)\right]_{-\pi/3}^{\pi/3}$$

$$= \left(\frac{1}{2}\left(\frac{\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right) - \left(\frac{1}{2}\left(-\frac{\pi}{3}\right) + \frac{1}{2}\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$= \left(\frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right) - \left(-\frac{\pi}{6} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right)$$

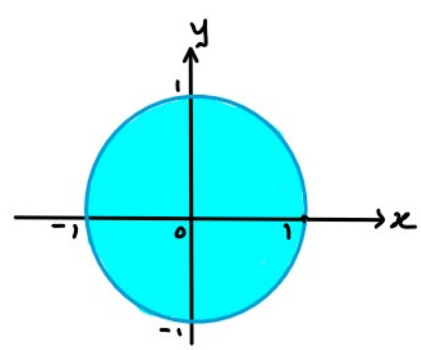
$$= \frac{\pi}{6} + \frac{\sqrt{3}}{4} + \frac{\pi}{6} + \frac{\sqrt{3}}{4}$$

$$= \frac{2\pi}{6} + \frac{2\sqrt{3}}{4}$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

19–27 Use polar coordinates to find the volume of the given solid.

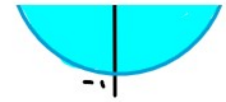
21. Below the plane $2x + y + z = 4$ and above the disk $x^2 + y^2 \leqslant 1$



In polar coordinates, the disk is described as

$D = \{(r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}.$

$D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$



Projection of the plane $2x+y+z=4$ on the disk $x^2+y^2 \le 1$

Also,

$$2x+y+z=4 \Rightarrow z = 4-2x-y.$$

Hence,

$$\text{Volume} = \iint_D (4-2x-y)\,dA$$

$$= \int_0^{2\pi}\int_0^1 (4-2r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (4r-2r^2\cos\theta - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[2r^2 - \frac{2}{3}r^3\cos\theta - \frac{r^3}{3}\sin\theta\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi}\left(2 - \frac{2}{3}\cos\theta - \frac{1}{3}\sin\theta\right)d\theta$$

$$= \left[2\theta + \frac{2}{3}\sin\theta - \frac{1}{3}\cos\theta\right]_0^{2\pi}$$

$$= \left(2(2\pi) + \frac{2}{3}\sin(2\pi) - \frac{1}{3}\cos(2\pi)\right) - \left(2(0) + \frac{2}{3}\sin 0 - \frac{1}{3}\cos 0\right)$$

$$= \left(4\pi + \frac{2}{3}(0) - \frac{1}{3}(1)\right) - \left(0 + \frac{2}{3}(0) - \frac{1}{3}(1)\right)$$

$$= 4\pi - \frac{1}{3} + \frac{1}{3}$$

$$= 4\pi.$$