Section 12.5: 4, 10, 19, 27, 51, 59, 71

2–5 Find a vector equation and parametric equations for the line.

4. The line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$

Here, a vector parallel to the line is $v = \langle 2, -3, 9 \rangle$.

Thus, a vector equation of the line through $(0, 14, -10) = r_0$

$$r = r_0 + tv$$

$$= \langle 0, 14, -10 \rangle + t\langle 2, -3, 9 \rangle$$

$$= \langle 2t, 14-3t, -10+9t \rangle$$

$$= 2t\,i + (14-3t)\,j + (-10+9t)\,k$$

So, letting $r = \langle x, y, z \rangle$, we get the parametric equation

$x = 2t$, $y = 14 - 3t$, $z = -10 + 9t$.

6–12 Find parametric equations and symmetric equations for the line.

10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$

Here, a vector parallel to the line is

$$v = (i + j) \times (j + k)$$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (1-0)i - (1-0)j + (1-0)k$$

$$= i - j + k$$

A vector equation of the line is

$$r = \langle 2,1,0\rangle + t\langle 1,-1,1\rangle$$

$$= \langle 2+t, 1-t, t\rangle$$

$$= (2+t)i + (1-t)j + tk$$

Thus, a parametric equation of the line is

$$x = 2+t, \quad y = 1-t, \quad z = t.$$

and a symmetric equation for the line is

$$x-2 = 1-y = z.$$ (by solving for t in each equation above)

19–22 Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19. L_1: $x = 3 + 2t, \quad y = 4 - t, \quad z = 1 + 3t$

L_2: $x = 1 + 4s, \quad y = 3 - 2s, \quad z = 4 + 5s$

Direction vector for L_1 $v_1 = \langle 2, -1, 1\rangle$.

Direction vector for L_2 $v_2 = \langle 4, -2, 5\rangle$.

If L_1 and L_2 are parall, then

$$v_1 = \alpha v_2 \text{ for some } \alpha \in \mathbb{R}.$$

$\Rightarrow \langle 2,-1,1\rangle = \alpha \langle 4,-2,5\rangle$

$\Rightarrow \quad 2 = 4\alpha, \quad -1 = -2\alpha, \quad 1 = 5\alpha$ (Impossible!!)

$\Rightarrow$ L_1 and L_2 are not parallel.

On the other hand, if L_1 and L_2 intersect, then we can find $s, t \in \mathbb{R}$ such that

$$x_1 = x_2, \quad y_1 = y_2 \quad \text{and} \quad z_1 = z_2.$$

i.e;

$$3+2t = 1+4s \quad \text{——— (1)}$$
$$4-t = 3-2s \quad \text{——— (2)}$$
$$1+3t = 4+5s \quad \text{——— (3)}$$

Solve simultaneously:

(1) + 2(2):

$$\begin{array}{rcl} 3+2t &=& 1+4s \\ 8-2t &=& 6-4s \\ \hline 11 &=& 7 \end{array}$$ (Impossible !!)

$\Rightarrow$ We cannot find s and t

$\Rightarrow$ L_1 and L_2 do not intersect.

Conclusion: since L_1 and L_2 are not parallel or intersect, it follows they are skew (don't lie on the same plane)

23–40 Find an equation of the plane.

27. The plane through the point $\underbrace{(1, -1, -1)}_{r_0}$ and parallel to the plane $5x - y - z = 6$

A normal vector to the plane $5x-y-z=6$ is $n = \langle 5,-1,-1\rangle$.
Since the required plane is parallel to this plane, a vector equation to the plane is

$$n \cdot (r - r_0) = 0$$

$$\Rightarrow \langle 5,-1,-1\rangle \cdot \left(\langle x,y,z\rangle - \langle 1,-1,-1\rangle\right) = 0$$

$$\Rightarrow \langle 5,-1,-1\rangle \cdot \langle x-1, y+1, z+1\rangle = 0$$

$$\Rightarrow 5(x-1) - 1(y+1) - 1(z+1) = 0$$

$$\Rightarrow 5x - 5 - y - 1 - z - 1 = 0$$

$$\Rightarrow 5x - y - z - 7 = 0$$

which is the required equation of the plane.

51–56 Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them. (Round to one decimal place.)

51. $x + 4y - 3z = 1, \quad -3x + 6y + 7z = 0$

Let n_1 be a normal vector to the plane $x+4y-3z=1$ and n_2, a normal to $-3x+6y+7z=0$.

Then

$$n_1 = \langle 1, 4, -3\rangle \text{ and } n_2 = \langle -3, 6, 7\rangle$$

If the two planes are parallel, then so are their normals.

But

$$n_1 = \alpha n_2$$

$$\Rightarrow \langle 1, 4, -3 \rangle = \alpha \langle -3, 6, 7 \rangle$$

$$\Rightarrow 1 = -3\alpha, \quad 4 = 6\alpha, \quad -3 = 7\alpha \quad \text{(Impossible)}$$

$\Rightarrow$ The planes are not parallel.

However,

$$n_1 \cdot n_2 = \langle 1, 4, -3 \rangle \cdot \langle -3, 6, 7 \rangle$$

$$= -3 + 24 - 21$$

$$= 0$$

$\Rightarrow$ The planes are orthogonal.

59–60 Find symmetric equations for the line of intersection of the planes.

59. $5x - 2y - 2z = 1, \quad 4x + y + z = 6$

Assuming the two planes intersect, then we can find $(x, y, z) \in \mathbb{R}^3$ that satisfies both equations above.

Since there are only two equations with three variables, we find (x, y, z) by trial and error.

Set $y = 0$. Then

$$5x - 2z = 1 \quad \text{———} \quad ①$$

$$4x + z = 6 \quad \text{———} \quad ②$$

$① + 2②$:

$$\begin{array}{rl} 5x - 2z &= 1 \\ 8x + 2z &= 12 \\ \hline 13x &= 13 \end{array}$$

$$\Rightarrow x = 1$$

Thus, $$4x+z=6$$
$$\Rightarrow z = 6-4x$$
$$= 6-4(1)$$
$$= 2.$$

So $(1,0,2)$ lies on the intersection of the two planes.

Since the line of intersection of the two planes lies on the two planes, we have this line to the orthogonal to the normals to the planes.

Normal to $5x-2y-2z=1$ is $n_1 = \langle 5,-2,-2\rangle$

Normal to $4x+y+z=6$ is $n_2 = \langle 4,1,1\rangle$

So a vector in the direction of the intersecting line is

$$v = n_1 \times n_2$$
$$= \begin{vmatrix} i & j & k \\ 5 & -2 & -2 \\ 4 & 1 & 1 \end{vmatrix}$$
$$= (-2+2)i - (5+8)j + (5+8)k$$
$$= \langle 0, -13, 13\rangle$$
$$= 13\langle 0,-1,1\rangle.$$

Thus, a vector equation of the line of intersection of the two planes is

$$r = r_0 + t\,v$$
$$= \langle 1,0,2\rangle + t\langle 0,-1,1\rangle$$
$$= \langle 1,-t,2+t\rangle$$

$\Rightarrow$ $x=1$, $y=-t$ and $z=2+t$ are the parametric equations

$\Rightarrow \quad x = 1, \; y = -t$ and $z = 2+t$ are the parametric equations

$\Rightarrow \quad x = 1, \quad -y = z-2$ are the symmetric equations.

71–72 Find the distance from the point to the given plane.

71. $(1, -2, 4), \quad 3x + 2y + 6z = 5$

By formula 9 in the book, the distance

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|3(1) + 2(-2) + 6(4) - 5|}{\sqrt{3^2+2^2+6^2}}$$

since $\langle a, b, c\rangle = \langle 3, 2, 6\rangle$ and $(x_1, y_1, z_1) = (1, -2, 4)$

$$= \frac{18}{\sqrt{49}}$$

$$= \frac{18}{7}$$