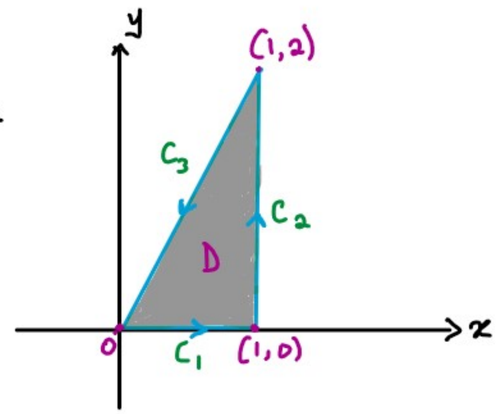
1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

3. $\oint_C xy\,dx + x^2y^3\,dy$,
C is the triangle with vertices (0, 0), (1, 0), and (1, 2)



ⓐ First we parametrize the lines. Recall the vector equation of a line going through r_0 to r_1 is

$$r = r_0 + t(r_1 - r_0), \quad 0 \le t \le 1$$

On C_1,

$r_0 = (0,0)$, $r_1 = (1,0)$.

$$\Rightarrow r = (0,0) + t(\langle 1-0, 0-0\rangle)$$

$$= (0,0) + \langle t, 0\rangle$$

$$= \langle t, 0\rangle.$$

$\Rightarrow x = t,\ y = 0$ are the parametric equations.

i.e.,

C_1: $x = t,\ y = 0,\ 0 \le t \le 1 \Rightarrow dx = dt,\ dy = 0dt$

On C_2,

$r_0 = (1,0)$, $r_1 = (1,2)$

$$\Rightarrow r = (1,0) + t(\langle 1-1, 2-0\rangle)$$

$$= (1,0) + t\langle 0, 2\rangle$$

$$= \langle 1, 2t\rangle$$

$\Rightarrow x = 1,\ y = 2t$ are the parametric equations i.e.,

C_2: $x = 1,\ y = 2t,\ 0 \le t \le 1 \implies dx = 0\,dt,\ dy = 2\,dt$

On C_3,

$$r_0 = (1,2),\ r_1 = (0,0)$$

$$\implies r = (1,2) + t(\langle 0-1, 0-2\rangle)$$

$$= (1,2) + \langle -t, -2t\rangle$$

$$= \langle 1-t, 2-2t\rangle.$$

$\implies x = 1-t,\ y = 2-2t$ are the parametric equations.

i.e.,

C_3: $x = 1-t,\ y = 2-2t,\ 0 \le t \le 1 \implies dx = -1\,dt,\ dy = -2\,dt$

Putting everything together with $C = C_1 + C_2 + C_3$,

$$\oint_C xy\,dx + x^2y^3\,dy = \int_{C_1} xy\,dx + x^2y^3\,dy + \int_{C_2} xy\,dx + x^2y^3\,dy + \int_{C_3} xy\,dx + x^2y^3\,dy$$

$$= \left(\int_0^1 t(0)\,dt + t^2(0)^3\,0\,dt\right) + \left(\int_0^1 1(2t)(0\,dt) + 1^2(2t)^3(2\,dt)\right) + \left(\int_0^1 (1-t)(2-2t)(-dt) + (1-t)^2(2-2t)^3(-2\,dt)\right)$$

$$= 0 + \int_0^1 16t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$$

$$= 4t^4\Big|_0^1 + \int_0^1 \left[-2(1-t)^2 - 16(1-t)^5\right] dt$$

$$= 4 - 2\int_0^1 (1-t)^2\,dt - 16\int_0^1 (1-t)^5\,dt$$

$u = 1-t \implies du = -dt$
$\implies dt = -du$

$$= 4 - 2\int_0^1 u^2(-du) - 16\int_0^1 u^5(-du)$$

$$= 4 + 2\left[\tfrac{1}{3}u^3\right]_{t=0}^{t=1} + 16\left[\tfrac{1}{6}u^6\right]_{t=0}^{t=1}$$

$$= 4 + \frac{2}{3}\Big[(1-t)^3\Big]_0^1 + \frac{16}{6}\Big[(1-t)^6\Big]_0^1$$

$$= 4 + \frac{2}{3}(0-1) + \frac{8}{3}(0-1)$$

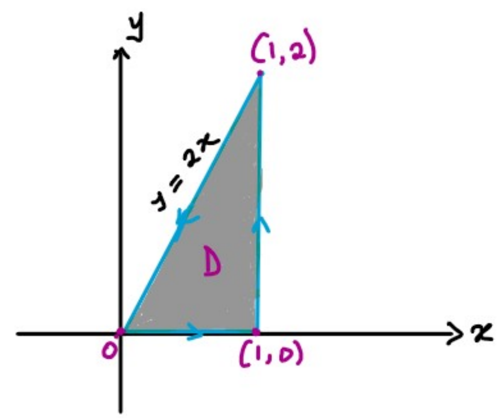
$$= 4 - \frac{2}{3} - \frac{8}{3}$$

$$= 4 - \frac{10}{3}$$

$$= \frac{12-10}{3}$$

$$= \frac{2}{3}.$$

(b) Using Green's Theorem,



$$m = \frac{2-0}{1-0} = 2$$

$$y = m(x-0)+0$$
$$= 2x$$

$$\oint_C \underbrace{xy}_{P}\,dx + \underbrace{x^2y^3}_{Q}\,dy = \iint_D \left[\frac{\partial}{\partial x}\underbrace{(x^2y^3)}_{Q} - \frac{\partial}{\partial y}\underbrace{(xy)}_{P}\right] dA$$

$$= \int_0^1\int_0^{2x}\left[2xy^3 - x\right] dy\,dx$$

$$= \int_0^1 \left[\frac{2}{4}xy^4 - xy\right]_{y=0}^{2x} dx$$

$$= \int_0^1 \left[\left(\frac{1}{2}x(2x)^4 - x(2x)\right) - (0-0)\right] dx$$

$$= \int_0^1 \left(8x^4 - 2x^2\right) dx$$

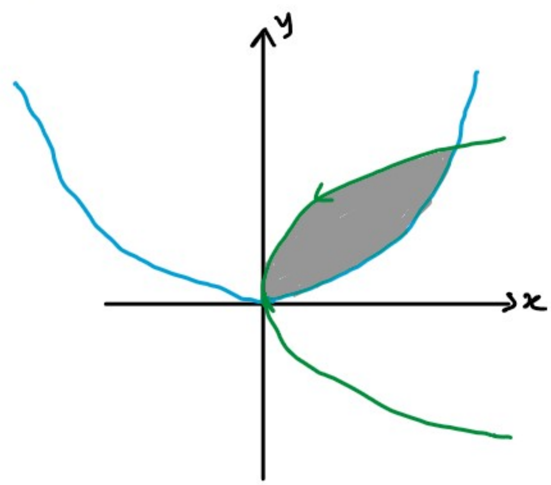
$$= \left[\frac{8}{6}x^5 - \frac{2}{3}x^3\right]_{x=0}^{x=1}$$

$$= \left(\frac{4}{3} - \frac{2}{3}\right) - (0 - 0)$$

$$= \frac{2}{3}.$$

5–10 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

7. $\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\, dy$,
C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$



Choosing positive orientation for C,

$$\oint_C \left(y + e^{\sqrt{x}}\right) dx + \left(2x + \cos y^2\right) dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(2x + \cos y^2\right) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2 - 1)\, dy\, dx$$

$$= \int_0^1 y \Big|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\sqrt{x} - x^2\right) dx$$

$$= \int_0^1 \left(x^{1/2} - x^2\right) dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0 - 0)$$

$y = x^2$, $x = y^2$

$\Rightarrow y = (y^2)^2 = y^4$

$\Rightarrow y(1 - y^3) = 0$

$\Rightarrow y = 0, y = 1$

$\Rightarrow x = 0, x = 1$

Curves intersect at $(0,0)$, $(1,1)$.

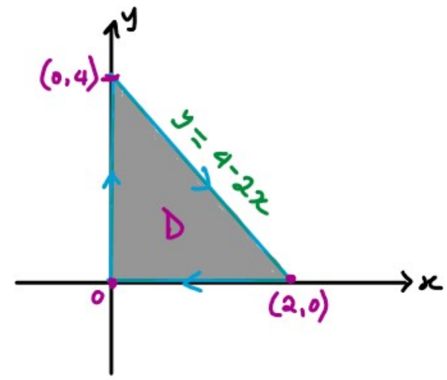
$$= \frac{1}{3}.$$

11–14 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

11. $\mathbf{F}(x, y) = \langle y \cos x - xy \sin x, xy + x \cos x \rangle$,
C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$

Since the curve C is traversed in the clockwise direction, a positive orientation is $-C$.

Thus,



$$m = \frac{4-0}{0-2} = -2$$

$$\Rightarrow y = m(x-2)+0 = -2x+4$$

$$\int_C F \cdot dr = -\int_{-C} P\,dx + Q\,dy$$

$$= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= -\iint_D \left[\frac{\partial}{\partial x}(xy + x\cos x) - \frac{\partial}{\partial y}(y\cos x - xy\sin x)\right] dA$$

$$= -\iint_D (y - x\sin x + \cos x - \cos x + x\sin x)\, dA$$

$$= -\iint_D y\, dA$$

$$= -\int_0^2 \int_0^{4-2x} y\, dy\,dx$$

$$= -\int_0^2 \frac{1}{2} y^2 \Big|_{y=0}^{y=4-2x} dx$$

$u = 4-2x \Rightarrow du = -2dx$

$$= -\frac{1}{2}\int_0^2 (4-2x)^2\,dx$$

$u = 4-2x \Rightarrow du = -2dx$

$\Rightarrow dx = \frac{du}{-2}$

$$= -\frac{1}{2}\int_0^2 u^2\cdot\left(\frac{du}{-2}\right)$$

$$= \frac{1}{4}\int_0^2 u^2\,du$$

$$= \frac{1}{4}\cdot\frac{1}{3}\,u^3\Big|_{x=0}^{x=2}$$

$$= \frac{1}{12}(4-2x)^3\Big|_0^2$$

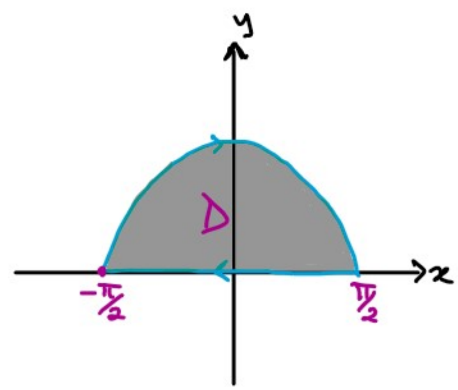
$$= \frac{1}{12}\left[(4-2\cdot 2)^3-(4-2\cdot 0)^3\right]$$

$$= \frac{1}{12}(0-4)^3$$

$$= -\frac{4\cdot 4\cdot 4}{12}$$

$$= -\frac{16}{3}.$$

12. $\mathbf{F}(x,y) = \langle e^{-x}+y^2, e^{-y}+x^2\rangle$,
C consists of the arc of the curve $y=\cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$



Again C is traversed clockwise, so

$$\oint_C F\cdot dr = -\int_{-C} P\,dx + Q\,dy$$

$$= -\iint\left[\frac{\partial}{\partial x}(e^{-y}+x^2) - \frac{\partial}{\partial y}(e^{-x}+y^2)\right]dA$$

$/2$ $\quad$ | $\quad$ $1/2$

$$= -\iint_D \left[\frac{\partial}{\partial x}\left(e^{-y}+x^2\right) - \frac{\partial}{\partial y}\left(e^{-x}+y^2\right)\right] dA$$

$$= -\iint_D (2x-2y)\, dA$$

$$= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos x} (2x-2y)\, dy\, dx$$

$$= -\int_{-\pi/2}^{\pi/2}\left[2xy - y^2\right]_{y=0}^{y=\cos x} dx$$

$$= -\int_{-\pi/2}^{\pi/2}\left(2x\cos x - \cos^2 x\right) dx$$

	u	dv
$+$	$2x$	$\cos x$
$-$	2	$\sin x$
$+$	0	$-\cos x$

$$= -\int_{-\pi/2}^{\pi/2} 2x\cos x\, dx + \int_{-\pi/2}^{\pi/2}\frac{1}{2}(1+\cos 2x)\, dx$$

$$= -\left[2x\sin x + 2\cos x\right]_{x=-\pi/2}^{\pi/2} + \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_{x=-\pi/2}^{\pi/2}$$

$$= -\left[2\left(\frac{\pi}{2}\right) - 2\left(-\frac{\pi}{2}\right)(-1)\right] + \left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right]$$

$$= -(\pi - \pi) + \left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}.$$