Section 14.8: 3, 5, 11, 33

3–14 Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2;\quad x^2 + y^2 = 1$

Let $g(x,y) = x^2+y^2-1=0$ Then

$\nabla f = \lambda \nabla g$

$\Longrightarrow (2x, -2y) = \lambda(2x, 2y)$

$\Rightarrow 2x = 2\lambda x$ and $-2y = 2\lambda y$

$\Rightarrow x(1-\lambda) = 0$ and $y(1+\lambda) = 0.$

$\Rightarrow x = 0$ or $\lambda = 1$ and $y(1+\lambda) = 0.$

If $x = 0$, then the constraint gives

$0^2 + y^2 = 1 \Rightarrow y = \pm 1$

If $\lambda = 1$, then the second condition gives

$y(1+1) = 0 \Rightarrow y = 0.$

Consequently, the constraint gives

$x^2 + 0^2 = 1 \Rightarrow x = \pm 1.$

Thus, possible points of extreme values are $(\pm 1, 0)$ and $(0, \pm 1)$.

Moreover,

$f(-1,0) = f(1,0) = 1-0 = 1$ and

$f(0,-1) = f(0,1) = 0-1 = -1$

Hence,

* the minimum value of f is $f(0, \pm 1) = -1$
* the maximum value of f is $f(\pm 1, 0) = 1$

given the constraint.

5. $f(x, y) = xy; \quad 4x^2 + y^2 = 8$

Let $g(x,y) = 4x^2 + y^2 - 8 = 0$ Then $\nabla f = \lambda \nabla g$ and $g(x,y) = 8$

$\Rightarrow (y, x) = \lambda (8x, 2y)$ and $g(x,y) = 8$

$\Rightarrow y = 8\lambda x$ and $x = 2\lambda y$ and $4x^2 + y^2 = 8$

Notice the three equations imply x or y cannot be zero.

Thus,

$$\frac{y}{8x} = \lambda = \frac{x}{2y} \quad \text{and} \quad 4x^2 + y^2 = 8$$

$$\Rightarrow 2y^2 = 8x^2 \quad \text{and} \quad 4x^2 + y^2 = 8$$

$$\Rightarrow y^2 = 4x^2 \quad \text{and} \quad 4x^2 + y^2 = 8$$

$$\Rightarrow 4x^2 + 4x^2 = 8$$

$$\Rightarrow 8x^2 = 8 \Rightarrow x = \pm 1$$

So $y^2 = 4x^2 \Rightarrow y^2 = 4(\pm 1)^2 \Rightarrow y = \pm 2$.

Thus, possible points of extreme values are $(\pm 1, \pm 2)$.

Furthermore,

$f(1,2) = xy\big|_{(1,2)} = 1(2) = 2$

$f(-1,2) = -1(2) = -2$

$f(1,-2) = 1(-2) = -2$

$f(-1,-2) = -1(-2) = 2$

Hence,

* f has the minimum value $f(-1,2) = -2 = f(1,-2)$ and

Hence,

* f has the minimum value $f(-1,2) = -2 = f(1,-2)$ and
* f has the maximum value $f(1,2) = 2 = f(-1,-2)$.

11. $f(x, y, z) = x^2 + y^2 + z^2; \quad x^4 + y^4 + z^4 = 1$

Let $g(x,y,z) = x^4 + y^4 + z^4 - 1 = 0$. Then $\nabla f = \lambda \nabla g$

$$\Rightarrow (2x, 2y, 2z) = \lambda (4x^3, 4y^3, 4z^3)$$

$$\Rightarrow 2x = 4\lambda x^3, \quad 2y = 4\lambda y^3, \quad 2z = 4\lambda z^3 \text{ and } g(x,y,z) = 1.$$

$$\Rightarrow 2x(1 - 2\lambda x^2) = 0, \quad 2y(1 - 2\lambda y^2) = 0, \quad 2z(1 - 2\lambda z^2) = 0 \text{ and } g(x,y,z) = 1$$

There are three cases for x, y, z that satisfy these four equations.

Case 1: All three variables are nonzero. i.e., $x \neq 0$ and $y \neq 0$ and $z \neq 0$.

$$\Rightarrow 1 - 2\lambda x^2 = 0, \quad 1 - 2\lambda y^2 = 0 \text{ and } 1 - 2\lambda z^2 = 0 \text{ and } g(x,y,z) = 1$$

$$\Rightarrow \lambda = \frac{1}{2x^2}, \quad \lambda = \frac{1}{2y^2}, \text{ and } \lambda = \frac{1}{2z^2} \text{ and } g(x,y,z) = 1$$

$$\Rightarrow \frac{1}{2x^2} = \frac{1}{2y^2} = \frac{1}{2z^2} \text{ and } g(x,y,z) = 1$$

$$\Rightarrow x^2 = y^2 = z^2 \text{ and } g(x,y,z) = 1$$

From plugging in $z^2 = x^2$ and $y^2 = x^2$, we obtain

$$x^4 + (x^2)^2 + (x^2)^2 = 1$$

$$\Rightarrow 3x^4 = 1 \Rightarrow x = \pm \frac{1}{\sqrt[4]{3}}$$

So $z^2 = x^2 \Rightarrow z = \pm \frac{1}{\sqrt[4]{3}}$ and

$$y^2 = x^2 \Rightarrow y = \pm \frac{1}{\sqrt[4]{3}}.$$

Thus we obtained many possible points as extreme value(s) for case 1, Though we obtained many possible points of extreme value(s) for case 1, fortunately,

$$f\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}\right) = \left(\pm\frac{1}{\sqrt[4]{3}}\right)^2 + \left(\pm\frac{1}{\sqrt[4]{3}}\right)^2 + \left(\pm\frac{1}{\sqrt[4]{3}}\right)^2$$

$$= \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}} = \sqrt{3}.$$

Case 2: Exactly one of the variables is zero.

i.e.; $\begin{cases} x = 0 & \text{but } y, z \neq 0 \\ y = 0 & \text{but } x, z \neq 0 \\ z = 0 & \text{but } x, y \neq 0 \end{cases}$

If $x = 0$ but $y, z \neq 0$, then

$$\left.\begin{aligned} 2y(1-2\lambda y^2) = 0 \Rightarrow 1 - 2\lambda y^2 = 0 \Rightarrow \lambda = \frac{1}{2y^2} \\ 2z(1-2\lambda z^2) = 0 \Rightarrow 1 - 2\lambda z^2 = 0 \Rightarrow \lambda = \frac{1}{2z^2} \end{aligned}\right\} \Rightarrow \frac{1}{2y^2} = \frac{1}{2z^2} \Rightarrow z^2 = y^2$$

Plugging into the constraint gives

$$0^4 + y^4 + (y^2)^2 = 1$$

$$\left.\begin{aligned} \Rightarrow 2y^4 = 1 \Rightarrow y = \pm\frac{1}{\sqrt[4]{2}} \\ \Rightarrow z = \pm\frac{1}{\sqrt[4]{2}} \end{aligned}\right\}$$

$\Rightarrow$ Possible extreme value points are $\left(0, \pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right)$

Similarly, for $y = 0$ but $x, z \neq 0$, we obtain the points $\left(\pm\frac{1}{\sqrt[4]{2}}, 0, \pm\frac{1}{\sqrt[4]{2}}\right)$

Similarly, for $y=0$ but $x, z \neq 0$, we obtain the points $\left(\pm\frac{1}{\sqrt[4]{2}}, 0, \pm\frac{1}{\sqrt[4]{2}}\right)$ and for $z=0$, $x, y \neq 0$, the point $\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}, 0\right)$

But

$$f\left(0, \pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right) = f\left(\pm\frac{1}{\sqrt[4]{2}}, 0, \pm\frac{1}{\sqrt[4]{2}}\right) = f\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}, 0\right)$$

$$= \left(\pm\frac{1}{\sqrt[4]{2}}\right)^2 + \left(\pm\frac{1}{\sqrt[4]{2}}\right)^2 + 0^2$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

Case 3: Exactly two of the variables are zero.

i.e., $\begin{cases} x=0, y=0 & \text{but } z \neq 0 \\ x=0, z=0 & \text{but } y \neq 0 \\ y=0, z=0 & \text{but } x \neq 0 \end{cases}$

For $x=0$, $y=0$ but $z \neq 0$, the constraint gives

$$0^4 + 0^4 + z^4 = 1 \Rightarrow z = \pm 1 \Rightarrow (0, 0, \pm 1)$$

For $x=0, z=0$ but $y \neq 0$, we obtain the points $(0, \pm 1, 0)$

For $y=0, z=0$ but $x \neq 0$, the points $(\pm 1, 0, 0)$.

Again,

$$f(0,0,\pm 1) = f(0, \pm 1, 0) = f(\pm 1, 0, 0)$$

$$= (\pm 1)^2 + 0^2 + 0^2$$

$$= (\pm 1)^2 + 0^2 + 0^2$$
$$= 1$$

Hence,

* the minimum value of f is 1 which occurs at the points found above
* the maximum value of f is $\sqrt{3}$, occurring at the points found above

for the given constraint.

31–43 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 14.7.

43. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point (4, 2, 0).

For any point (x, y, z) on the cone, the distance of it from $(4,2,0)$ is

$$d = \sqrt{(x-4)^2 + (y-2)^2 + (z-0)^2}$$

$$\Rightarrow d^2 = (x-4)^2 + (y-2)^2 + z^2.$$

Thus, problem is equivalent to

minimize $f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$ Subject to $z^2 = x^2 + y^2$.

Let $g(x,y,z) = x^2 + y^2 - z^2 = 0.$

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow (2(x-4), 2(y-2), 2z) = \lambda(2x, 2y, -2z)$$

$$\Rightarrow 2(x-4) = 2\lambda x, \quad 2(y-2) = 2\lambda y, \quad 2z = -2\lambda z \text{ and } x^2 + y^2 = z^2$$

$$\Rightarrow x - 4 = \lambda x \quad y - 2 = \lambda y, \quad z = -\lambda z \text{ and } x^2 + y^2 = z^2$$

$\Rightarrow \quad x-4 = \lambda x \qquad y-2=\lambda y \;, \; z = -\lambda z$ and $x^2+y^2 = z^2$

$\Rightarrow \quad x(1-\lambda) = 4, \quad y(1-\lambda) = 2 \;, \; z(1+\lambda) = 0$ and $x^2+y^2 = z^2$

$\Rightarrow \quad z=0$ or $\lambda = -1$

$z=0$ is not possible since $x^2+y^2=z^2 \Rightarrow x^2+y^2=0 \Rightarrow x=y=0$ which do not satisfy the other equations.

So $\lambda = -1$.

$\Rightarrow \quad x(1+1) = 4 \;, \; y(1+1) = 2 \;, \quad z^2 = x^2+y^2$

$\Rightarrow \quad x = 2, \quad y = 1 \;, \quad z^2 = 2^2+1^2 = 5$

$\Rightarrow \quad x=2, \; y=1, \quad z = \pm\sqrt{5}$

Hence, $(2,1,\pm\sqrt{5})$ are the points on the cone closest to $(4,2,0)$.