5–22 Find the limit, if it exists, or show that the limit does not exist.

9. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$

Let $y = mx$. Then we approach $(0,0)$ along this line.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2} = \lim_{x\to 0} \frac{x^4 - 4(mx)^2}{x^2 + 2(mx)^2}$$

$$= \lim_{x\to 0} \frac{x^2(x^2 - 4m^2)}{x^2(1 + 2m^2)}$$

$$= \lim_{x\to 0} \frac{x^2 - 4m^2}{1 + 2m^2}$$

$$= \frac{-4m^2}{1 + 2m^2}$$

Since the limit depends on the slope of the line, we will get different limits for different lines through the origin.

E.g when $m = 1$, ie; $y = x$,

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2} = \left.\frac{-4m^2}{1 + 2m^2}\right|_{m=1} = \frac{-4}{3}$$

when $m = 0$, ie; $y = 0$,

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2} = \left.\frac{-4m^2}{1 + 2m^2}\right|_{m=0} = 0$$

18. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{xy^4}{x^2 + y^8}$

$y = mx$ as follows

$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

Let $y = mx$. Then we approach $(0,0)$ along the line $y = mx$ as follows

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = \lim_{x\to 0} \frac{x(mx)^4}{x^2+(mx)^8}$$

$$= \lim_{x\to 0} \frac{m^4x^5}{x^2(1+m^8x^6)}$$

$$= \lim_{x\to 0} \frac{m^4x^3}{1+m^8x^6}$$

$$= 0$$

So the limit exists for all lines.

After several trial and error, we found that for $x = y^4$,

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = \lim_{y\to 0} \frac{y^4 \cdot y^4}{y^8+y^8}$$

$$= \lim_{y\to 0} \frac{y^8}{2y^8}$$

$$= \frac{1}{2}$$

Hence, the limit does not exist.

25–26 Find $h(x, y) = g(f(x, y))$ and the set of points at which h is continuous.

25. $g(t) = t^2 + \sqrt{t}$, $\quad f(x, y) = 2x + 3y - 6$

$$h(x,y) = g(f(x,y))$$

$$= g(2x+3y-6)$$

$$= (2x+3y-6)^2 + \sqrt{2x+3y-6}$$

$$= (2x+3y-6)^2 + \sqrt{2x+3y-6}$$

This is continuous if $2x+3y-6 \geqslant 0$ since f is continuous for $x, y \in \mathbb{R}$.

Hence, h is continuous on the set

$$\{(x,y) \in \mathbb{R}^2 : y \geqslant 2 - \tfrac{2}{3}x\}.$$

29–38 Determine the set of points at which the function is continuous.

29. $F(x, y) = \dfrac{xy}{1 + e^{x-y}}$

The functions xy and $1+e^{x-y}$ are defined for all $x, y \in \mathbb{R}$ and $1+e^{x-y}$ is never zero for all $x, y \in \mathbb{R}$.

Hence, F is continuous in $\mathbb{R}^2$. i.e. F is continuous everywhere.

37. $f(x, y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

We suspect f might be discontinuous only at $(x,y) = (0,0)$ since f is defined everywhere except at $(x,y) = (0,0)$.

So we compute $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ to check. Here, we use the idea of the squeeze theorem.

Notice that

$$x^2 \leq 2x^2 + y^2 \quad \text{for all } x, y \in \mathbb{R}$$

$$\Longrightarrow \quad \frac{x^2}{2x^2+y^2} \leq 1 \quad \text{for all } x, y \neq 0.$$

$$\Rightarrow \quad \left| \frac{x^2y^3}{} \right| \leq |y^3| \quad \text{for all } x, y \neq 0.$$

$$\Rightarrow \quad \left|\frac{x^2y^3}{2x^2+y^2}\right| \le |y^3| \quad \text{for all } x, y \neq 0.$$

$$\Rightarrow \quad \lim_{(x,y)\to(0,0)} \left|\frac{x^2y^3}{2x^2+y^2}\right| \le \lim_{y\to 0} |y^3| = 0$$

Thus, by Squeeze theorem

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{2x^2+y^2} = 0 \neq 1 = f(0,0).$$

Hence, f is continuous everywhere except at $(x,y) = (0,0)$.

i.e., f is continuous on the set

$$\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}.$$

39–41 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geqslant 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

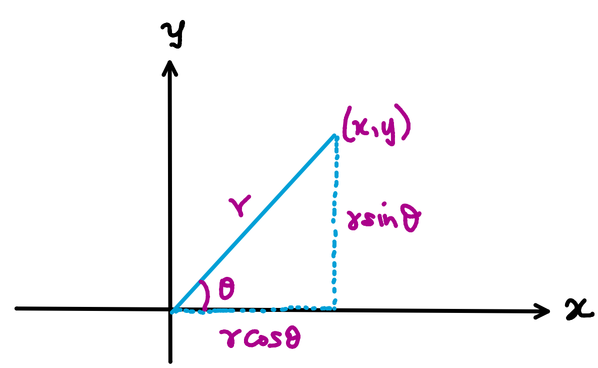
39. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^3 + (r\sin\theta)^3}{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \lim_{r\to 0^+} \frac{r^2[r\cos^3\theta + r\sin^3\theta]}{r^2[\cos^2\theta + \sin^2\theta]}$$

$$= \lim_{r\to 0^+} (r\cos^3\theta + r\sin^3\theta)$$

$$= 0$$



$x = r\cos\theta$ and $y = r\sin\theta$ and notice that $(x,y) \to (0,0)$ along the blue line $\Rightarrow r \to 0^+$