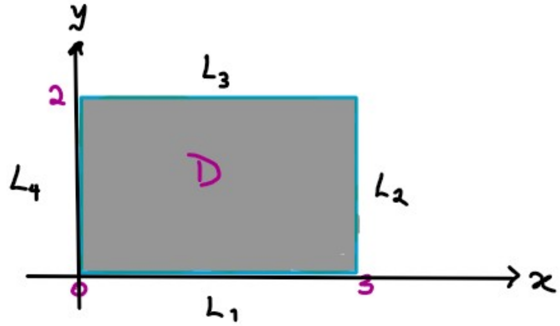
EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

f has absolute max and min since f is continuous and D is closed and bounded.

The critical points will satisfy $\nabla f(x,y) = 0$.



$\Rightarrow (2x-2y, -2x+2) = (0,0)$.

$\Rightarrow 2x-2y = 0$ and $-2x+2=0$

$\Rightarrow x = 1$ and $2(1)-2y=0$

$\Rightarrow x=1$ and $y=1$

$\Rightarrow (1,1)$ is the only critical point.

$\Rightarrow f(1,1) = 1^2 - 2(1)(1) + 2(1) = 1$

By an extension of the Closed Interval Method, we need to compute the max and min on the lines L_1, L_2, L_3, L_4, the boundaries of D.

On L_1, $y = 0$.

$\Rightarrow f(x,0) = x^2, \quad 0 \le x \le 3$

$\Rightarrow f_x(x,0) = 2x > 0$ on $0 < x \le 3 \Rightarrow f \nearrow$

$\Rightarrow f(0,0) = 0$ is the min and $f(3,0) = 3^2 - 2(3)(0) + 2(0) = 9$, the max on L_1.

On L_2, $x = 3$.

$\Rightarrow f(3,y) = 9 - 6y + 2y = 9 - 4y, \quad 0 \le y \le 2$

$\Rightarrow f_y(3,y) = -4 < 0 \Rightarrow f \searrow$

$\Rightarrow f(3,0) = 9$ is the max and $f(3,2) = 3^2 - 2(3)(2) + 2(2) = 1$, the min on L_2.

On L_3, $y = 2$.

$\Rightarrow f(x,2) = x^2 - 4x + 4, \quad 0 \le x \le 3$

$\Rightarrow f_x(x,2) = 2x - 4 = 0 \Rightarrow x = 2$ is a critical point of $f(x,2)$

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So $\left.\begin{array}{l} f(0,2) = 4 \\ f(2,2) = 0 \\ f(3,2) = 1 \end{array}\right\} \Rightarrow (2,2)$ is min point while $(0,2)$, the max point.

Also, on L_4, $x = 0$.

$\Rightarrow f(0,y) = 2y, \quad 0 \le y \le 2.$

$f_y(0,y) = 2 > 0 \Rightarrow f \nearrow$

$\Rightarrow f(0,0) = 0$ is min and $f(0,2) = 4$ is max points.

Summarizing the steps above, we have

$f(1,1) = 1 \qquad f(2,2) = 0$

$f(0,0) = 0 \qquad f(0,2) = 4$

$f(3,0) = 9 \qquad f(3,2) = 1$

Hence, the absolute maximum value of f is 9 and it occurs at $(3,0)$ and the absolute minimum value of f is 0 and it occurs at $(0,0)$ and $(2,2)$.

5–20 Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

11. $f(x, y) = x^3 - 3x + 3xy^2$

To find the critical points, we solve for x and y such that $\nabla f(x,y) = (0,0)$

i.e.,

$$(3x^2 - 3 + 3y^2,\ 6xy) = (0,0).$$

$\Rightarrow 6xy = 0$ and $3x^2 - 3 + 3y^2 = 0$

$\Rightarrow \quad x=0$ or $y=0$ and $3x^2-3+3y^2=0$.

$x=0 \Rightarrow -3+3y^2=0 \Rightarrow y=\pm 1$

and $\quad y=0 \Rightarrow 3x^2-3=0 \Rightarrow x=\pm 1$

So the critical points are $(0,\pm 1)$, $(\pm 1, 0)$

We now use the Second Derivative Test

$$D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

to completely characterise the points.

$f_x = 3x^2-3+3y^2 \Rightarrow f_{xx} = 6x$ and $f_{xy} = 6y$

$f_y = 6xy \Rightarrow f_{yy} = 6x$

Thus

$$\begin{aligned} D(0,1) &= f_{xx}(0,1)\, f_{yy}(0,1) - [f_{xy}(0,1)]^2 \\ &= 0\,(6) - [6]^2 \\ &= -36 < 0 \Rightarrow (0,1) \text{ is a saddle point.} \end{aligned}$$

Compute $D(0,-1) = ?$ classify the point $(0,-1)$.

$$\begin{aligned} D(1,0) &= f_{xx}(1,0)\, f_{yy}(1,0) - [f_{xy}]^2 \\ &= 6\,(6) - [0]^2 \\ &= 36. \end{aligned}$$

and $f_{xx}(1,0) = 6 > 0 \Rightarrow f(1,0) = -2$ is a local minimum value.

Compute $D(-1,0) = ?$ Classify the point $(-1,0)$.

20. $f(x,y) = \sin x \sin y, \quad -\pi < x < \pi, \quad -\pi < y < \pi$

At critical point, $\nabla f(x,y) = 0$.

$\Rightarrow (\cos x \sin y,\ \sin x \cos y) = (0,0)$.

At critical point, $\nabla f(x,y) = 0$.

$\Rightarrow (\cos x \sin y,\ \sin x \cos y) = (0,0)$.

$\Rightarrow \cos x \sin y = 0$ and $\sin x \cos y = 0$

$\Rightarrow \cos x = 0$ or $\sin y = 0$ and $\sin x = 0$ or $\cos y = 0$

$\cos x = 0 \Rightarrow x = \pm \frac{\pi}{2}$ since $-\pi < x < \pi$

$\sin y = 0 \Rightarrow y = 0$ since $-\pi < y < \pi$.

So,

$x = \pm\frac{\pi}{2} \Rightarrow \sin\left(\pm\frac{\pi}{2}\right)\cos y = 0 \Rightarrow \cos y = 0 \Rightarrow y = \pm\frac{\pi}{2}$.

and

$y = 0 \Rightarrow \sin x \cos(0) = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0$

Thus $\left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$, $(0,0)$ are the critical numbers.

Furthermore,

$f_{xx}(x,y) = -\sin x \sin y$, $f_{xy} = \cos x \cos y$

$f_{yy}(x,y) = -\sin x \sin y$

Note that

$$\begin{aligned} D(a,b) &= f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2 \\ &= (-\sin a \sin b)(-\sin a \sin b) - [\cos a \cos b]^2 \\ &= (\sin a \sin b)^2 - (\cos a \cos b)^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow D(0,0) &= (\sin 0 \sin 0)^2 - (\cos 0 \cos 0)^2 \\ &= -1 < 0 \end{aligned}$$

$\Rightarrow (0,0)$ is a saddle point.

$$\begin{aligned} D\left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right) &= \left[\sin\left(\pm\frac{\pi}{2}\right)\sin\left(\pm\frac{\pi}{2}\right)\right]^2 - \left[\cos\left(\pm\frac{\pi}{2}\right)\cos\left(\pm\frac{\pi}{2}\right)\right]^2 \\ &= 1 - 0 \\ &= 1 > 0 \end{aligned}$$

$f_{xx}\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) = -\sin\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) = -1(-1)(-1) = -1 < 0$ $\Rightarrow \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$ is a local max pt.

$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a local max point

$f_{xx}(-\frac{\pi}{2},-\frac{\pi}{2}) = -\sin(-\frac{\pi}{2})\sin(-\frac{\pi}{2}) = -1(-1)(-1) = -1 < 0 \Rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ ...

$f_{xx}(\frac{\pi}{2},\frac{\pi}{2}) = -\sin(\pi/2)\sin(\pi/2) = -1 < 0 \Rightarrow$ $(\pi/2, \pi/2)$ is a local max point

$f_{xx}(-\frac{\pi}{2},\frac{\pi}{2}) = -\sin(-\frac{\pi}{2})\sin(\frac{\pi}{2}) = 1 > 0$

$f_{xx}(\frac{\pi}{2},-\frac{\pi}{2}) = -\sin(\pi/2)\sin(-\pi/2) = 1 > 0$

$\Rightarrow$ $(-\pi/2, \pi/2)$ and $(\pi/2, -\pi/2)$ are local min points

Hence

$f(-\frac{\pi}{2},-\frac{\pi}{2}) = f(\frac{\pi}{2},\frac{\pi}{2}) = 1$ is a local maximum value

$f(-\pi/2,\pi/2) = f(\pi/2,-\pi/2) = 1$ is a local minimum value.

31–38 Find the absolute maximum and minimum values of f on the set D.

33. $f(x, y) = x^2 + y^2 + x^2y + 4$,
$D = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 1\}$

$\nabla f(x,y) = 0$

$\Rightarrow (2x + 2xy, 2y + x^2) = (0,0)$

$\Rightarrow 2x(1+y) = 0$ and $2y + x^2 = 0$

$\Rightarrow x = 0$ or $y = -1$ and $2y + x^2 = 0$

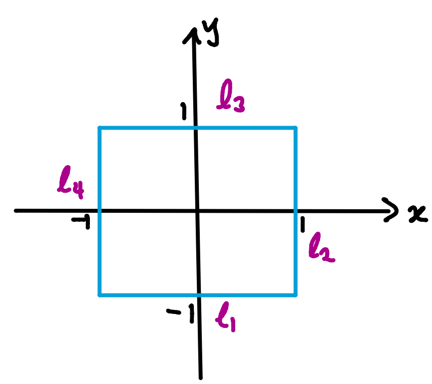
$x = 0 \Rightarrow 2y + 0^2 = 0 \Rightarrow y = 0$

$y = -1 \Rightarrow 2(-1) + x^2 = 0 \Rightarrow x = \pm\sqrt{2}$.

So $(0,0)$ is the only critical number in D since $-1 \leq x \leq 1$

and $f(0,0) = 4$.

Next, we find the optimal points on l_1, l_2, l_3, l_4.



On l_1, $y = -1$.

$$\Rightarrow f(x,-1) = x^2 + (-1)^2 + x^2(-1) + 4$$
$$= x^2 + 1 - x^2 + 4$$
$$= 5$$

$\Rightarrow f(x,-1) = 5 \quad \text{for all} \quad -1 \le x \le 1$

On l_2, $x = 1$.

$\Rightarrow f(1,y) = 1^2 + y^2 + 1^2(y) + 4$

$= y^2 + y + 5$

$\Rightarrow f_y(1,y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ is a critical number

$\Rightarrow f(1,-1) = 1^2 + (-1)^2 + (1)^2(-1) + 4 = 5$

$f(1,-\frac{1}{2}) = 1^2 + (-\frac{1}{2})^2 + 1^2(-\frac{1}{2}) + 4 = \frac{19}{4}$

$f(1,1) = 1^2 + 1^2 + 1^2(1) + 4 = 7$

On l_3, $y = 1$.

$\Rightarrow f(x,1) = x^2 + 1^2 + x^2(1) + 4$

$= 2x^2 + 5$

$\Rightarrow f_x(x,1) = 4x = 0 \Rightarrow x = 0$ is a critical number.

$\Rightarrow f(-1,1) = (-1)^2 + 1^2 + (-1)^2(1) + 4 = 7$

$f(0,1) = 0^2 + 1^2 + 0^2(1) + 4 = 5$

$f(1,1) = 7$ as computed already.

On l_4, $x = -1$.

$\Rightarrow f(-1,y) = (-1)^2 + y^2 + (-1)^2 y + 4$

$= y^2 - y + 5$

$\Rightarrow f_y(-1,y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$ is a critical number.

$\Rightarrow f(-1,-1) = (-1)^2 + (-1)^2 + (-1)^2(-1) + 4 = 5$

$f(-1,\frac{1}{2}) = (-1)^2 + (\frac{1}{2})^2 + (-1)^2(\frac{1}{2}) + 4 = \frac{23}{4}$

$f(-1,1) = 7$ as computed already.

Comparing everything in green, we obtain

$f(\pm 1, 1) = 7$ is the absolute maximum value and

$f(0,0) = 4$, the absolute minimum value.

43. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

Let (x, y, z) be any point on the cone. Then we are looking for the argument that makes

$$d\left((x,y,z), (4,2,0)\right) = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$

the smallest.

Since the minimizer of d lies on the cone,

$$d\left((x,y,z), (4,2,0)\right) = \sqrt{(x-4)^2 + (y-2)^2 + x^2 + y^2}$$

$$\Rightarrow d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 =: f(x,y).$$

Since $d \geq 0$, d^2 is monotone. So minimizing d is equivalent to minimizing f.

$$\nabla f(x,y) = 0$$

$$\Rightarrow \left(2(x-4) + 2x,\ 2(y-2) + 2y\right) = (0,0)$$

$$\Rightarrow 2x - 8 + 2x = 0 \text{ and } 2y - 4 + 2y = 0$$

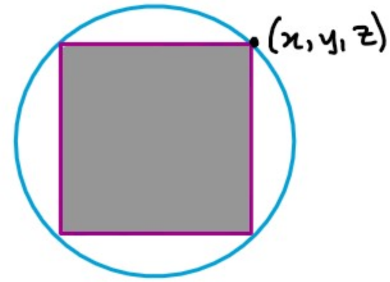
$$\Rightarrow x = 2 \text{ and } y = y = 1$$

$$\Rightarrow z^2 = 2^2 + 1^2 = 3 \Rightarrow z = \pm\sqrt{5}.$$

So the critical points are $(2, 1, \pm\sqrt{5})$ and these are the points on the cone closest to the point $(4, 2, 0)$.

47. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

If the box and sphere are centered at the origin, and the edges of the box align to the x, y and z axes, then the figure shows a typical cross section through the inscription with the sphere given by $x^2+y^2+z^2=r^2$. The box can then be described by



length $l = 2y$

width $w = 2x$

height $h = 2z$

where $0<x, y, z<r$ (ie; (x,y,z) lies in the first octant)

$$\Rightarrow \text{Volume } V = lwh$$
$$= 8xyz.$$
$$V(x,y) = 8xy\sqrt{r^2-x^2-y^2}$$
$$= 8xy\left(r^2-x^2-y^2\right)^{1/2}$$

Our goal is to maximize $V(x,y)$.

ie; find (x,y) s.t. $\nabla V(x,y) = (V_x, V_y) = (0,0)$.

But

$$V_x = 8y\left(r^2-x^2-y^2\right)^{1/2} + 4xy\left(r^2-x^2-y^2\right)^{-1/2}(-2x)$$
$$= 8y\sqrt{r^2-x^2-y^2} - \frac{8x^2y}{\sqrt{r^2-x^2-y^2}}$$
$$= \frac{8y\left(r^2-x^2-y^2\right) - 8x^2y}{\sqrt{r^2-x^2-y^2}}$$
$$= \frac{8r^2y - 8x^2y - 8y^3 - 8x^2y}{\sqrt{r^2-x^2-y^2}}$$
$$= \frac{8r^2y - 16x^2y - 8y^3}{}$$

$$= \frac{8r^2y - 16x^2y - 8y^3}{\sqrt{r^2-x^2-y^2}}$$

$$= \frac{8y\left(r^2-2x^2-y^2\right)}{\sqrt{r^2-x^2-y^2}}$$

and

$$V_y = 8x\left(r^2-x^2-y^2\right)^{1/2} + 4xy\left(r^2-x^2-y^2\right)^{-1/2}(-2y)$$

$$= 8x\sqrt{r^2-x^2-y^2} - \frac{8xy^2}{\sqrt{r^2-x^2-y^2}}$$

$$= \frac{8x\left(r^2-x^2-y^2\right) - 8xy^2}{\sqrt{r^2-x^2-y^2}}$$

$$= \frac{8r^2x - 8x^3 - 8xy^2 - 8xy^2}{\sqrt{r^2-x^2-y^2}}$$

$$= \frac{8r^2x - 8x^3 - 16xy^2}{\sqrt{r^2-x^2-y^2}}$$

$$= \frac{8x\left(r^2-x^2-2y^2\right)}{\sqrt{r^2-x^2-y^2}}$$

So

$$V_x = 0 \Rightarrow \frac{8y\left(r^2-2x^2-y^2\right)}{\sqrt{r^2-x^2-y^2}} = 0$$

$$\Rightarrow 8y\left(r^2-2x^2-y^2\right) = 0$$

$$\Rightarrow r^2-2x^2-y^2 = 0 \quad \text{since } y>0$$

$$\Rightarrow 2x^2+y^2 = r^2 \quad \text{———} \quad (1)$$

$$\Rightarrow \quad r^2 - 2x^2 - y^2 = 0 \quad \text{since } y \ldots$$

$$\Rightarrow \quad 2x^2 + y^2 = r^2 \quad \text{——————} \quad (1)$$

and

$$V_y = 0 \Rightarrow \frac{8x\left(r^2 - x^2 - 2y^2\right)}{\sqrt{r^2 - x^2 - y^2}} = 0$$

$$\Rightarrow 8x\left(r^2 - x^2 - 2y^2\right) = 0$$

$$\Rightarrow r^2 - x^2 - 2y^2 = 0 \quad \text{Since } x > 0$$

$$\Rightarrow x^2 + 2y^2 = r^2 \quad \text{——————} \quad (2)$$

$2 \times (1) - (2)$:

$$\begin{aligned} 4x^2 + 2y^2 &= 2r^2 \\ x^2 + 2y^2 &= r^2 \\ \hline 3x^2 &= r^2 \\ \Rightarrow x &= \frac{r}{\sqrt{3}}. \end{aligned}$$

Similarly

$2 \times (2) - (1)$:

$$\begin{aligned} 2x^2 + 4y^2 &= 2r^2 \\ 2x^2 + y^2 &= r^2 \\ \hline 3y^2 &= r^2 \\ \Rightarrow y &= \frac{r}{\sqrt{3}}. \end{aligned}$$

Hence, the maximum volume of the rectangle is

$$V\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = 8xy\sqrt{r^2 - x^2 - y^2}\,\Big|_{\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)}$$

$$= 8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{3}}\right)^2 - \left(\frac{r}{\sqrt{3}}\right)^2}$$

$$= \frac{8r^2}{\ } \sqrt{r^2 - \frac{r^2}{\ } - \frac{r^2}{\ }}$$

$$= \frac{8r^2}{3} \sqrt{r^2 - \frac{r^2}{3} - \frac{r^2}{3}}$$

$$= \frac{8r^2}{3} \sqrt{\frac{r^2}{3}}$$

$$= \frac{8r^3}{3\sqrt{3}}.$$