Section 13.4: 3, 13, 23

**3–8** Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of $t$.

**3.** $\mathbf{r}(t) = \langle -\frac{1}{2}t^2, t \rangle, \quad t = 2$

Velocity

$$v(t) = r'(t)$$
$$= \langle -t, 1 \rangle.$$

$\Rightarrow$ Speed $= |v(t)|$

$$= \sqrt{(-t)^2 + 1^2}$$
$$= \sqrt{t^2 + 1}$$

Acceleration

$$a(t) = v'(t) = r''(t)$$
$$= \langle -1, 0 \rangle, \text{ uniform acceleration}$$

At $t = 2$,

$$r(2) = \langle -\tfrac{1}{2}(2)^2, 2 \rangle = \langle -2, 2 \rangle.$$

$$v(2) = \langle -2, 1 \rangle$$

$$a(2) = \langle -1, 0 \rangle$$

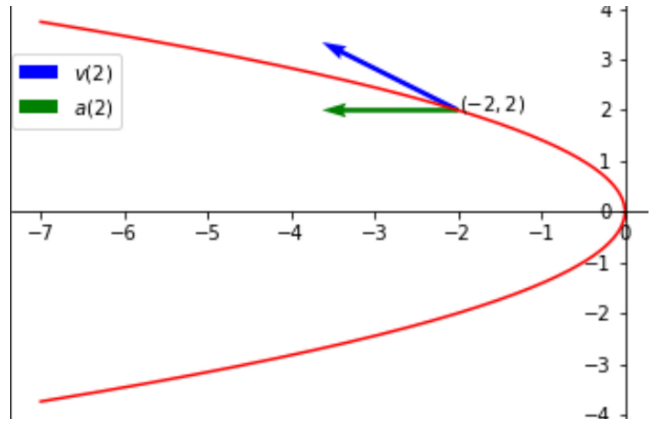
To sketch the path, we rewrite in xy coordinates.

$$\langle x, y\rangle = \mathbf{r} = \langle -\tfrac{1}{2}t^2, t\rangle$$

$$\Rightarrow y = t$$

Thus,

$$x = -\tfrac{1}{2}y^2.$$




**9–14** Find the velocity, acceleration, and speed of a particle with the given position function.

**13.** $\mathbf{r}(t) = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k})$

The velocity

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

$$= e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}) + e^t(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})$$

$$= e^t((\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + (t+1)\mathbf{k})$$

The acceleration

$$\mathbf{a}(t) = \mathbf{v}'(t)$$

$$= e^t[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + (t+1)\mathbf{k}] + e^t[(-\sin t - \cos t)\mathbf{i} + (\cos t - \sin t)\mathbf{j} + \mathbf{k}]$$

$$= e^t[-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + (t+2)\mathbf{k}]$$

The speed

$$= |\mathbf{v}(t)|$$

The speed

$$= |v(t)|$$

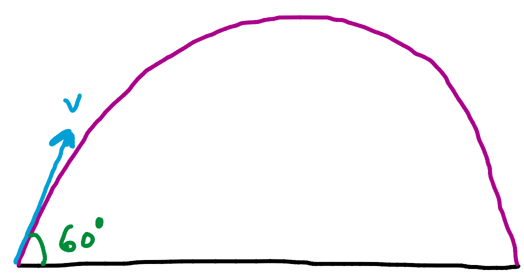
$$= \sqrt{e^{2t}\left[(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + (t+1)^2\right]}$$

$$= e^t \sqrt{\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\cos t \sin t + \cos^2 t + t^2 + 2t + 1}$$

$$= e^t \sqrt{t^2 + 2t + 3}$$

**23.** A projectile is fired with an initial speed of 200 m/s and angle of elevation 60°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

As given $|v(0)| = 200 m/s$ at an angle of $60°$




$$\Rightarrow v(0) = 200\cos 60° i + 200 \sin 60° j$$

$$= 100 i + 100\sqrt{3} j$$

But $|a| \approx 9.8 m/s^2$

$$\Rightarrow a = -9.8 j$$ since acceleration is due to gravity

$$\Rightarrow v(t) = \int -9.8 j \, dt$$

$$= -9.8t j + c$$

So

$$100 i + 100\sqrt{3} j = v(0) = 0 + c$$

$$\Longrightarrow v(t) = 100 i + (100\sqrt{3} - 9.8t) j$$

Thus

$$r(t) = \int v(t)\,dt$$

$$= \int \left[100i + (100\sqrt{3} - 9.8t)j\right] dt$$

$$= 100ti + (100\sqrt{3}t - 4.9t^2)j + D$$

We assume the projectile is fired from the ground, so that $r(0) = 0$.

$\Rightarrow D = 0$

$\Rightarrow r(t) = 100ti + (100\sqrt{3}t - 4.9t^2)j$

$\Rightarrow x(t) = 100t$, $y(t) = 100\sqrt{3}t - 4.9t^2$ are the parametric equations of the projectile.

(a) Range of the projectile is the horizontal distance the projectile travels.

So $y = 100\sqrt{3}t - 4.9t^2 = 0$

$$\Rightarrow t(100\sqrt{3} - 4.9t) = 0$$

$$\Rightarrow t = 0, \quad t = \frac{100\sqrt{3}}{4.9}$$

$\Rightarrow x(0) = 0$ and $x\left(\frac{100\sqrt{3}}{4.9}\right) = 100\left(\frac{100\sqrt{3}}{4.9}\right) \approx 3534.7976$

So the range $\approx 3534m$

(b) At maximum height $y'(t) = 0$

$$\Rightarrow 100\sqrt{3} - 9.8t = 0$$

$$\Rightarrow t = \frac{100\sqrt{3}}{9.8}$$

Thus, the maximum height is

$$y\left(\frac{100\sqrt{3}}{9.8}\right) = 100\sqrt{3}\left(\frac{100\sqrt{3}}{9.8}\right) - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2$$

$$\approx 1531m$$

(c) From part (a), impact occurs at $t = \frac{100\sqrt{3}}{4.9}$

$$\Rightarrow v\left(\frac{100\sqrt{3}}{4.9}\right) = 100i + \left[100\sqrt{3} - 9.8\left(\frac{100\sqrt{3}}{4.9}\right)\right]j$$

$$= 100i - 100\sqrt{3}j$$

$\Rightarrow$ speed at impact is

$$\left|v\left(\frac{100\sqrt{3}}{4.9}\right)\right| = \sqrt{100^2 + (-100\sqrt{3})^2}$$

$$= \sqrt{10000 + 30000}$$

$$= \sqrt{40000}$$

$$= 200 m/s$$