19–26 Find a parametric representation for the surface.

19. The plane through the origin that contains the vectors $\mathbf{i} - \mathbf{j}$ and $\mathbf{j} - \mathbf{k}$

The required vector equation is

$$\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b} \quad \text{where } u, v \text{ are real numbers.}$$

Here,

$$\mathbf{r}_0 = (0,0,0), \quad \mathbf{a} = \langle 1,-1,0\rangle, \quad \mathbf{b} = \langle 0,1,-1\rangle.$$

Thus,

$$\mathbf{r} = (0,0,0) + u\langle 1,-1,0\rangle + v\langle 0,1,-1\rangle.$$

$$= (0+u+0,\ 0-u+v,\ 0+0-v).$$

$$= (u,\ v-u,\ -v).$$

Hence, the parametric equations are

$$x = u, \quad y = v-u, \quad z = -v.$$

33–36 Find an equation of the tangent plane to the given parametric surface at the specified point.

35. $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$; $\quad u = 1,\ v = \pi/3$

At $u=1$, $v=\pi/3$, a point on the tangent plane is

$$r(1, \pi/3) = (1\cos\pi/3,\ 1\sin\pi/3,\ \pi/3)$$

$$= \left(\tfrac{1}{2},\ \tfrac{\sqrt{3}}{2},\ \pi/3\right).$$

At any u, v normal to the tangent plane is $r_u \times r_v$.

But $\quad (\cos v, \sin v, 0)$ and $r_v = (-u\sin v, u\cos v, 1)$

But $r_u = (\cos v, \sin v, 0)$ and $r_v = (-u\sin v, u\cos v, 1)$

$$\Rightarrow r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= (\sin v)i - (\cos v)j + (u\cos^2 v + u\sin^2 v)k$$

$$= \langle \sin v, -\cos v, u \rangle.$$

$$\Rightarrow r_u(1, \pi/3) \times r_v(1, \pi/3) = \langle \sin \pi/3, -\cos \pi/3, 1 \rangle$$

$$= \langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 1 \rangle.$$

Hence, the equation of tangent plane at $(\frac{1}{2}, \frac{\sqrt{3}}{2}, \pi/3)$ is

$$\frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + 1\left(z - \pi/3\right) = 0.$$

$$\Rightarrow \frac{1}{2}\left(\sqrt{3}x - y + 2z\right) = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} + \frac{\pi}{3}$$

$$\Rightarrow \frac{1}{2}\left(\sqrt{3}x - y + 2z\right) = \pi/3.$$

39–50 Find the area of the surface.

43. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2}),\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1$

Let x and y be the parameters. Then the vector equation of the surface is

$$r(x, y) = xi + yj + \frac{2}{3}(x^{3/2} + y^{3/2})k$$

is

$$r(x,y) = xi + yj + \tfrac{2}{3}\left(x^{3/2} + y^{3/2}\right)k$$

$$\Rightarrow r_x = i + x^{1/2}k \quad \text{and} \quad r_y = j + y^{1/2}k$$

$$\Rightarrow r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & x^{1/2} \\ 0 & 1 & y^{1/2} \end{vmatrix}$$

$$= -x^{1/2}i - y^{1/2}j + k$$

$$\Rightarrow |r_x \times r_y| = \sqrt{\left(-x^{1/2}\right)^2 + \left(-y^{1/2}\right)^2 + 1^2}$$

$$= \sqrt{x + y + 1}$$

Thus, area of the surface

$$= \iint_D |r_x \times r_y|\, dA \qquad \text{where } D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$$

$$= \int_0^1 \int_0^1 \sqrt{x+y+1}\, dx\, dy$$

$$= \int_0^1 \int_0^1 (x+y+1)^{1/2}\, dx\, dy \qquad u = x+y+1 \Rightarrow du = dx$$

$$= \int_0^1 \int_0^1 u^{1/2}\, du\, dy$$

$$= \int_0^1 \frac{2}{3} u^{3/2}\Big|_{x=0}^{x=1} dy$$

$x=0$

$$= \int_0^1 \frac{2}{3}(x+y+1)^{3/2}\Big|_{x=0}^{x=1} dy$$

$$= \frac{2}{3}\int_0^1 \left[(2+y)^{3/2} - (y+1)^{3/2}\right] dy$$

$$= \frac{2}{3}\left[\frac{2}{5}(2+y)^{5/2} - \frac{2}{5}(1+y)^{5/2}\right]_0^1$$

$$= \frac{2}{3}\left[\left(\frac{2}{5}(3^{5/2}) - \frac{2}{5}(2^{5/2})\right) - \left(\frac{2}{5}(2^{5/2}) - \frac{2}{5}(1^{5/2})\right)\right]$$

$$= \frac{2}{3}\cdot\frac{2}{5}\left(3^{5/2} - 2\cdot 2^{5/2} + 1\right)$$

$$= \frac{4}{15}\left(3^{5/2} - 2^{7/2} + 1\right).$$