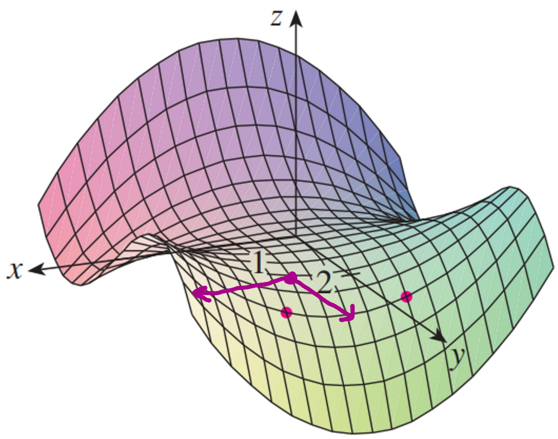
Section 14.3: 5, 11, 15, 20, 23, 28, 42, 47, 53

5–8 Determine the signs of the partial derivatives for the function f whose graph is shown.



5. (a) $f_x(1, 2)$ (b) $f_y(1, 2)$

As shown in purple on the graph, from the point $(1,2)$ and

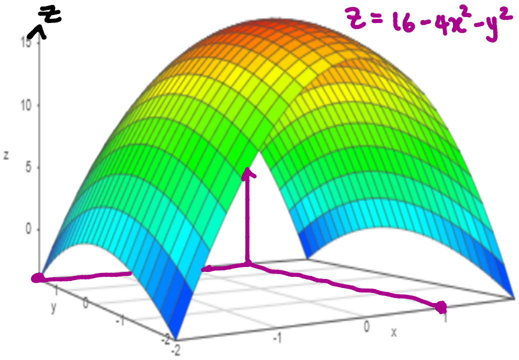
ⓐ moving in the positive x-axis, f is increasing, so $f_x(1,2) > 0$.

ⓑ moving in the positive y-axis, f is decreasing, so $f_x(1,2) < 0$.

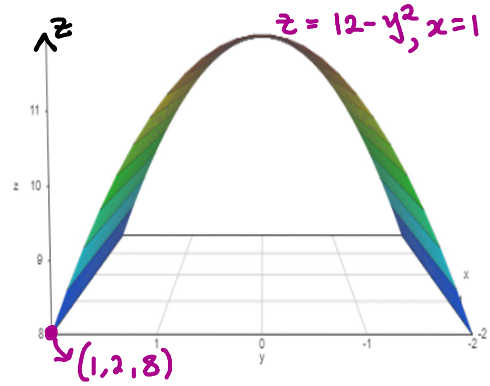
11. If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

$$f_x(x,y) = -8x \Rightarrow f_x(1,2) = -8x\Big|_{(x,y)=(1,2)} = -8(1) = -8$$

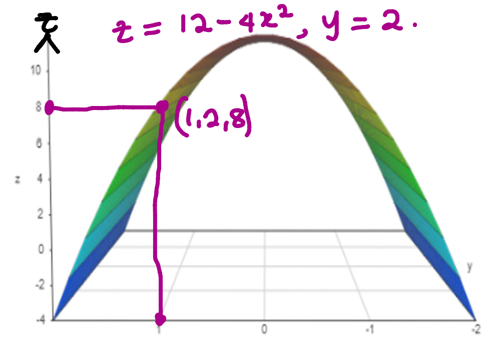
$$f_y(x,y) = -2y \Rightarrow f_y(1,2) = -2y\Big|_{(x,y)=(1,2)} = -2(2) = -4$$



$f(x,y)$



$f_x(1,2)$



$f_y(1,2)$

15–40 Find the first partial derivatives of the function.

15. $f(x, y) = x^4 + 5xy^3$

$f_x(x,y) = 4x^3 + 5y^3$, Keeping y fixed.

$f_y(x,y) = 15xy^2$, Keeping x fixed.

20. $z = x\sin(xy)$

Using product rule and keeping y fixed,

$z_x = x\,(y\cos(xy)) + 1\,(\sin(xy))$

$= xy\cos(xy) + \sin(xy)$

Similarly keeping x fixed (now, no product rule)

$z_y = x\,(x\cos(xy))$

$= x^2\cos(xy)$

23. $f(x, y) = \dfrac{ax + by}{cx + dy}$

Keeping y fixed and using quotient rule,

$$f_x(x,y) = \frac{(cx+dy)a - (ax+by)c}{(cx+dy)^2}$$

$$= \frac{acx + ady - acx - bcy}{(cx+dy)^2}.$$

$$= \frac{(ad-bc)y}{(cx+dy)^2}$$

Similarly, keeping x fixed and using quotient rule,

$$f_y(x,y) = \frac{(cx+dy)b - (ax+by)d}{(cx+dy)^2}$$

$$= \frac{bcx + bdy - adx - bdy}{(cx+dy)^2}$$

$$= \frac{(bc-ad)x}{(cx+dy)^2}$$

28. $f(x, y) = x^y$

Here, we use implicit differentiation as follows.

Let $z = x^y$.

Then $\ln z = \ln x^y = y \ln x$.

ie;

$$\ln z = y \ln x.$$

i.e.,

$$\ln z = y \ln x.$$

Keeping y fixed and differentiating implicitly w.r.t x,

$$\frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(y \ln x)$$

$$\Rightarrow \quad \frac{1}{z} z_x = y\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad z_x = \frac{yz}{x}$$

$$= \frac{y(x^y)}{x}$$

$$\Rightarrow \quad f_x(x,y) = y(x^{y-1})$$

Similly, keeping x fixed and differentiating w.r.t y,

$$\frac{\partial}{\partial y}(\ln z) = \frac{\partial}{\partial y}(y \ln x)$$

$$\Rightarrow \quad \frac{1}{z} z_y = \ln x$$

$$\Rightarrow \quad z_y = z \ln x$$

$$= x^y \ln x$$

$$\Rightarrow \quad f_y(x,y) = x^y \ln x.$$

Work out these ones yourself

24. $w = \dfrac{e^v}{u + v^2}$

30. $F(\alpha, \beta) = \int_\alpha^\beta \sqrt{t^3 + 1}\, dt$

41–44 Find the indicated partial derivative.

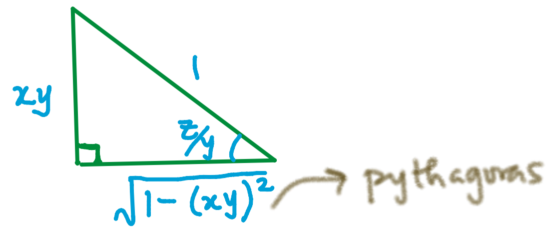
42. $f(x, y) = y \sin^{-1}(xy)$; $f_y(1, \frac{1}{2})$

Here, we use implicit differentiation as follows.

Let $z = y \sin^{-1}(xy)$.

Then

$$\frac{z}{y} = \sin^{-1}(xy)$$



$$\Rightarrow \quad \sin\left(\frac{z}{y}\right) = xy.$$

Keeping x fixed and differentiating implicitly w.r.t y,

$$\frac{\partial}{\partial x}\left(\sin\left(\underbrace{\frac{z}{y}}_{\text{q-rule}}\right)\right) = \frac{\partial}{\partial x}(xy)$$

Keep in mind $z = f(x,y)$, a function of x & y

$$\cos\left(\frac{z}{y}\right) \cdot \frac{y z_y - z}{y^2} = x.$$

$$\Rightarrow \quad y z_y - z = \frac{xy^2}{\cos\left(\frac{z}{y}\right)}$$

$$\Rightarrow \quad y z_y = \frac{xy^2}{\cos(z/y)} + z$$

$$\Rightarrow \quad z_y = \frac{xy}{\cos(z/y)} + \frac{z}{y}$$

$$= \frac{xy}{\frac{\sqrt{1-(xy)^2}}{1}} + \frac{z}{y}$$

$$= \frac{xy}{\sqrt{1-(xy)^2}} + \frac{z}{y}$$

So

$$f_y(x,y) = \frac{xy}{\sqrt{1-(xy)^2}} + \sin^{-1}(xy)$$

Hence,

$$f_y\left(1,\tfrac{1}{2}\right) = \frac{1\left(\frac{1}{2}\right)}{\sqrt{1-\left(1\cdot\frac{1}{2}\right)^2}} + \sin^{-1}\left(1\cdot\tfrac{1}{2}\right)$$

$$= \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} + \sin^{-1}\left(\tfrac{1}{2}\right)$$

$$= \frac{1/2}{\sqrt{3}/2} + \frac{\pi}{6}$$

$$= \frac{1}{\sqrt{3}} + \frac{\pi}{6}$$

47–50 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

47. $x^2 + 2y^2 + 3z^2 = 1$

Keeping y fixed and differentiating implicitly w.r.t x,

$$2x + 6z \cdot z_x = 0$$

$$\Rightarrow z_x = \frac{-2x}{6z}$$

$$= \frac{-x}{3z}$$

Similarly, Keeping x fixed and differentiating implicitly w.r.t y,

$$4y + 6z \cdot z_y = 0$$

$$\Rightarrow z_y = \frac{-4y}{6z}$$

$$= \frac{-2y}{3z}$$

53–58 Find all the second partial derivatives.

53. $f(x, y) = x^4 y - 2x^3 y^2$

$$f_x(x,y) = 4x^3 y - 6x^2 y^2$$

$$\Rightarrow f_{xx}(x,y) = 12x^2 y - 12xy^2 \quad \text{and} \quad \boxed{f_{xy}(x,y) = 4x^3 - 12x^2 y}$$

Notice they are same.

Also,

$$f_y(x,y) = x^4 - 4x^3 y$$

$$\Rightarrow f_{yy}(x,y) = -4x^3 \quad \text{and} \quad \boxed{f_{yx} = 4x^3 - 12x^2 y}$$