Section 12.3: 7, 9, 17, 23, 39, 51

2–10 Find $\mathbf{a} \cdot \mathbf{b}$.

7. $\mathbf{a} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$a \cdot b = (2i + j) \cdot (i - j + k)$$
$$= 2(1) + 1(-1) + 0(1)$$
$$= 2 - 1 + 0$$
$$= \underline{\underline{1}}$$

9. $|\mathbf{a}| = 7, \quad |\mathbf{b}| = 4, \quad$ the angle between $\mathbf{a}$ and $\mathbf{b}$ is $30°$

By Theorem 3 in the book,

$$a \cdot b = |a||b| \cos\theta$$ where θ is the angle between a and b.

Thus

$$a \cdot b = 7(4)\cos 30°$$
$$= 7(4)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \underline{\underline{14\sqrt{3}}}$$

15–20 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

17. $\mathbf{a} = \langle 1, -4, 1 \rangle, \quad \mathbf{b} = \langle 0, 2, -2 \rangle$

By Theorem 3 in the book,

$$a \cdot b = |a||b| \cos\theta$$ where θ is the angle between a and b.

$$\Rightarrow \cos\theta = \frac{a \cdot b}{|a||b|}$$

$$= \frac{\langle 1,-4,1\rangle \cdot \langle 0,2,-2\rangle}{|\langle 1,-4,1\rangle| \, |\langle 0,2,-2\rangle|}$$

$$= \frac{1(0) - 4(2) + 1(-2)}{\sqrt{1^2+(-4)^2+1^2}\,\sqrt{0^2+2^2+(-2)^2}}$$

$$= \frac{-8-2}{\sqrt{18}\,\sqrt{8}}$$

$$= \frac{-10}{12}$$

$$= -\frac{5}{6}$$

$\Rightarrow \theta = \cos^{-1}\left(-5/6\right) \approx 146^\circ$ to the nearest degree.

23–24 Determine whether the given vectors are orthogonal, parallel, or neither.

23. (a) $\mathbf{a} = \langle 9, 3\rangle$, $\mathbf{b} = \langle -2, 6\rangle$

Recall two vectors a and b are

- orthogonal if $a \cdot b = 0$
- parallel if $a = \alpha b$ for some $\alpha \in \mathbb{R}$.

Thus,

$$a \cdot b = \langle 9,3\rangle \cdot \langle -2,6\rangle = 9(-2) + 3(6) = -18 + 18 = 0$$

$\Rightarrow$ a and b are orthogonal.

(b) $\mathbf{a} = \langle 4, 5, -2\rangle$, $\mathbf{b} = \langle 3, -1, 5\rangle$

$$\begin{aligned} a \cdot b &= \langle 4,5,-2\rangle \cdot \langle 3,-1,5\rangle \\ &= 4(3)+5(-1)-2(5) \\ &= 12-5-10 \\ &= -3 \neq 0 \end{aligned}$$

$\Rightarrow$ a and b are not orthogonal

On the other hand, suppose a and b are parallel. Then we can find $\alpha \in \mathbb{R}$ such that

$$\langle 4,5,-2\rangle = \alpha\langle 3,-1,5\rangle.$$

$$\Rightarrow \quad 4 = 3\alpha, \quad 5 = -\alpha, \quad -2 = 5\alpha \quad \text{(Impossible!!)}$$

Hence, the two vectors are neither orthogonal nor parallel.

(c) $\mathbf{a} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$

$$\begin{aligned} a \cdot b &= (-8i+12j+4k)\cdot(6i-9j-3k) \\ &= -8(6)+12(-9)+4(-3) \\ &= -168 \neq 0 \end{aligned}$$

$\Rightarrow$ a and b are not orthogonal.

Suppose a and b are parallel. Then we can find $\alpha \in \mathbb{R}$ such that

$$(-8i+12j+4k) = \alpha(6i-9j-3k)$$

$$\Rightarrow \quad -(8+6\alpha)i + (12+9\alpha)j + (4+3\alpha)k = 0$$

$$\Rightarrow \quad -(8+6\alpha) = 0, \quad 12+9\alpha = 0, \quad 4+3\alpha = 0$$

$$\Rightarrow \quad \alpha = -\frac{4}{3}, \quad \alpha = -\frac{4}{3}, \quad \alpha = \frac{-4}{3}$$

$\Rightarrow$ a and b are parallel.

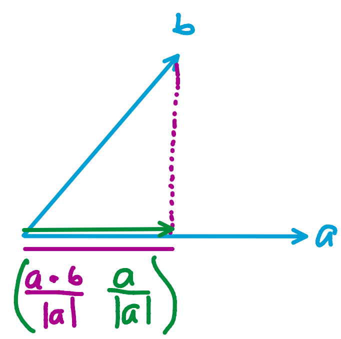
(d) $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 5\mathbf{i} + 9\mathbf{j} - 2\mathbf{k}$

$$\begin{aligned} a \cdot b &= (3i - j + 3k) \cdot (5i + 9j - 2k) \\ &= 3(5) - 1(9) + 3(-2) \\ &= 15 - 9 - 6 \\ &= 0 \end{aligned}$$

$\Rightarrow$ a and b are orthogonal.

39–44 Find the scalar and vector projections of **b** onto **a**.

39. $\mathbf{a} = \langle -5, 12 \rangle, \quad \mathbf{b} = \langle 4, 6 \rangle$



The scalar projection of b onto a

$$\begin{aligned} \text{comp}_a b &= \frac{a \cdot b}{|a|} \\ &= \frac{\langle -5, 12 \rangle \cdot \langle 4, 6 \rangle}{\sqrt{(-5)^2 + 12^2}} \\ &= \frac{-5(4) + 12(6)}{\sqrt{169}} \\ &= \frac{52}{13} \\ &= \underline{\underline{4}} \end{aligned}$$

The vector projection of b onto a

$$\text{Proj}_a b = \left(\frac{a \cdot b}{|a|} \frac{a}{|a|}\right)$$

$$= 4 \frac{a}{|a|}$$

$$= \frac{4 \langle -5, 12 \rangle}{\sqrt{(-5)^2 + 12^2}}$$

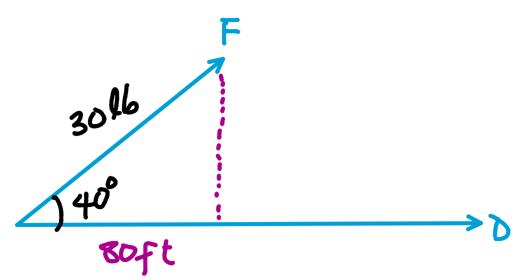
$$= \frac{4 \langle -5, 12 \rangle}{13}$$

$$= \left\langle \frac{-20}{13}, \frac{48}{13} \right\rangle.$$

51. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of 40° above the horizontal moves the sled 80 ft. Find the work done by the force.

As given,

$|F| = 30\,lb$, $|D| = 80ft$, $\theta = 40°$



Work done

$$W = F \cdot D$$

$$= |F|\,|D| \cos\theta$$

$$= 30(80) \cos 40°$$

$$\approx 1838.5067 \text{ ft-lb}$$

1–7 Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

1. $\mathbf{a} = \langle 2, 3, 0 \rangle, \quad \mathbf{b} = \langle 1, 0, 5 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{vmatrix} = i\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} - j\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} + k\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 15i - 10j - 3k$$

$$a \cdot (a \times b) = (2i + 3j + 0k) \cdot (15i - 10j - 3k)$$
$$= 2(15) + 3(-10) + 0(-3)$$
$$= 30 - 30 - 0$$
$$= 0$$

and

$$b \cdot (a \times b) = (i + 0j + 5k)(15i - 10j - 3k)$$
$$= 1(15) + 0(-10) + 5(-3)$$
$$= 15 - 0 - 15$$
$$= 0.$$

$\Rightarrow$ The cross product $a \times b$ is orthogonal to both a and b.

9–12 Find the vector, not with determinants, but by using properties of cross products.

9. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$ **10.** $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$

10

$$k \times (i - 2j) = (k \times i) + (k \times -2j) \qquad \text{By property 3}$$

$$= (k \times i) - 2(k \times j) \qquad \text{By property 2}$$

$$= j \quad -2(\; -i \;)$$

NB: $k = 0i + 0j + k$
$i = i + 0j + 0k$
$j = 0i + j + 0k$

$$= j + 2i$$

$$= 2i + j$$

35–36 Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.

35. $P(-2, 1, 0)$, $Q(2, 3, 2)$, $R(1, 4, -1)$, $S(3, 6, 1)$

$$\vec{PQ} = (2,3,2) - (-2,1,0) = \langle 4, 2, 2 \rangle = a$$

$$\vec{PR} = (1,4,-1) - (-2,1,0) = \langle 3, 3, -1 \rangle = b$$

$$\vec{PS} = (3,6,1) - (-2,1,0) = \langle 5, 5, 1 \rangle = c$$

Thus,

$$a \times (b \times c) = \begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix}$$

$$= 4(3+5) - 2(3+5) + 2(15-15)$$

$$= 32 - 16$$

$$= 16$$

Hence, the volume of the parallelpiped with adjacent edges $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ is 16 cubic units.