Section 16.3: 4, 13, 17

3–10 Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

4. $\mathbf{F}(x, y) = (y^2 - 2x)\,\mathbf{i} + 2xy\,\mathbf{j}$

Here, we apply Theorem 6.

Domain of F is $\mathbb{R}^2$ which is open and simply connected.

Moreover,

$$P = y^2 - 2x \implies \frac{\partial P}{\partial y} = 2y$$

$$Q = 2xy \implies \frac{\partial Q}{\partial x} = 2y$$

i.e. P and Q have continuous first partial derivatives and

$$\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x}$$

Hence, F is conservative by Theorem 6.

Now, we find f such that $\nabla f = F$.

i.e.,

$$\frac{\partial f}{\partial x} = P = y^2 - 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = Q = 2xy$$

$$\implies f(x,y) = \int (y^2 - 2x)\,\partial x$$

$$= xy^2 - x^2 + g(y)$$

Differentiating wrt y and comparing with Q gives

$$\frac{\partial f}{\partial y} = 2xy + g'(y) = 2xy$$

$$\implies g'(y) = 0 \implies g(y) = c, \text{ a constant.}$$

Hence, the required function is

Hence, the required function is

$$f(x,y) = xy^2 - x^2 + C.$$

12–18 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

13. $\mathbf{F}(x, y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$,
C: $\mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t \rangle, \quad 0 \leqslant t \leqslant 1$

(a) $\nabla f = F$

$$\Rightarrow \frac{\partial f}{\partial x} = x^2y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^3y^2.$$

$$\Rightarrow f(x,y) = \int x^2y^3\,\partial x$$

$$= \frac{1}{3}x^3y^3 + g(y)$$

Differentiating wrt y and using the second equation gives

$$\frac{\partial f}{\partial y} = x^3y^2 + g'(y) = x^3y^2$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c, \text{ a constant.}$$

Thus, the required function is

$$f(x,y) = \frac{1}{3}x^3y^3 + c$$

(b) By the Fundamental Theorem for line Integrals (Theorem 2),

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr$$

$$= f(r(b)) - f(r(a)) \qquad a = 0 \text{ and } b = 1$$

$$= f(r(b)) - f(r(a))$$

$$= f(-1,3) - f(0,0)$$

$$= \frac{1}{3}(-1)^3(3)^3 + c - \left[\frac{1}{3}(0)^3(0)^3 + c\right]$$

$$= -9.$$

$a = 0$ and $b = 1$

$\Rightarrow r(0) = \langle 0^3 - 2(0),\ 0^3 + 2(0)\rangle$

$= \langle 0, 0\rangle$

and

$r(1) = \langle 1^3 - 2(1),\ 1^3 + 2(1)\rangle$

$= \langle -1, 3\rangle$

17. $\mathbf{F}(x, y, z) = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$,
C: $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} + (t^2 - 2t)\,\mathbf{k}$,
$0 \leqslant t \leqslant 2$

(a) $\nabla f = F$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial x} = yze^{xz}}_{\mathrm{I}},\quad \underbrace{\frac{\partial f}{\partial y} = e^{xz}}_{\mathrm{II}} \text{ and } \underbrace{\frac{\partial f}{\partial z} = xye^{xz}}_{\mathrm{III}}$$

(I) implies

$$f(x,y,z) = \int yze^{xz}\,\partial x$$

$$= ye^{xz} + g(y,z)$$

Differentiating and equating to (II) gives

$$\frac{\partial f}{\partial y} = e^{xz} + g_y(y,z) = e^{xz}$$

$$\Rightarrow g_y(y,z) = 0$$

$$\Rightarrow g(y,z) = \int 0\,\partial y$$

$$= h(z).$$

So far,

$$f(x,y,z) = ye^{xz} + h(z)$$

Again, differentiating and equating to III gives

$$\frac{\partial f}{\partial z} = xye^{xz} + h'(z) = xye^{xz}$$

$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = c.$$

Hence,

$$f(x,y,z) = ye^{xz} + c.$$

6 $r(0) = \langle 0^2+1,\ 0^2-1,\ 0^2-2(0)\rangle = \langle 1,-1,0\rangle$ and

$r(2) = \langle 2^2+1,\ 2^2-1,\ 2^2-2(2)\rangle = \langle 5,3,0\rangle$

So by the Fundamental Theorem for line integrals,

$$\int_C F\cdot dr = \int \nabla f\cdot dr$$

$$= f(r(b)) - f(r(a))$$

$$= f(r(2)) - f(r(0))$$

$$= f(5,3,0) - f(1,-1,0)$$

$$= (3e^{5(0)} + c) - (-1e^{1(0)} + c)$$

$$= 3 + c + 1 - c$$

$$= 4.$$