Section 15.7: 7, 17, 23, 30

9–10 Write the equations in cylindrical coordinates.

9. (a) $x^2 - x + y^2 + z^2 = 1$ (b) $z = x^2 - y^2$

(a) $x^2 - x + y^2 + z^2 = 1 \Rightarrow x^2 + y^2 - x + z^2 = 1$

$\Rightarrow r^2 - r\cos\theta + z^2 = 1$ since $x^2+y^2 = r^2$ and $x = r\cos\theta$

$\Rightarrow z^2 = 1 + r\cos\theta - r^2$
in cylindrical coordinates

(b) $z = x^2 - y^2 \Rightarrow z = (r\cos\theta)^2 - (r\sin\theta)^2$

$= r^2\cos^2\theta - r^2\sin^2\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$

$= r^2\cos(2\theta)$

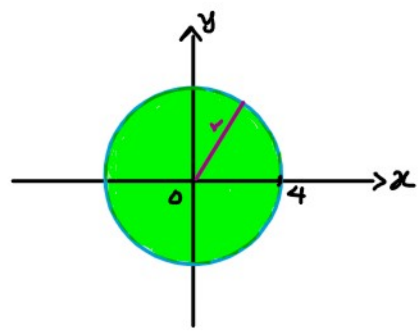
17–28 Use cylindrical coordinates.

17. Evaluate $\iiint_E \sqrt{x^2+y^2}\, dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

The region E is described as

$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 4,\ -5 \le z \le 4\}.$

Thus,



Projection of the cylinder on x-y plane

$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\, r\, dz\, dr\, d\theta$$

$$= \int^{2\pi}\int^4 r^2 [z]^4 \, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^4 r^2 \left[z \right]_{-5}^{4} \, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^4 9r^2 \, dr\, d\theta$$

$$= \int_0^{2\pi} \left[3r^3 \right]_0^4 \, d\theta$$

$$= \int_0^{2\pi} 3(4)^3 \, d\theta$$

$$= 192 \int_0^{2\pi} d\theta$$

$$= 192 \left[\theta \right]_0^{2\pi}$$

$$= 192(2\pi)$$

$$= 384\pi.$$

23. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.

In cylindrical coordinates,

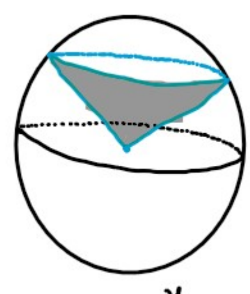
$$z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$$

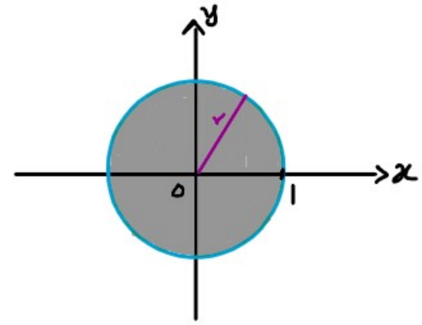
and

$$x^2+y^2+z^2 = 2 \Rightarrow r^2 + z^2 = 2$$

$$\Rightarrow z^2 = 2 - r^2.$$

$$\Rightarrow z = \sqrt{2-r^2}$$





The surfaces intersect when

$$r^2 = z^2 = 2 - r^2$$

$\Rightarrow \quad r^2 = 2 - r^2$

$\Rightarrow \quad 2r^2 = 2$

$\Rightarrow \quad r = \pm 1$

$\Rightarrow \quad r = 1$ since $r > 0$.

Thus, the solid is desirbed by the region

$$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r \le z \le \sqrt{2 - r^2}\}$$

Hence, the volume of the solid

$$V = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r z \Big|_{z=r}^{z=\sqrt{2-r^2}} dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r \left(\sqrt{2-r^2} - r\right) dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r\sqrt{2-r^2} - r^2\right) dr\, d\theta$$

$\int r\sqrt{2-r^2}\,dr$ let $u = 2 - r^2$

$du = -2r\,dr \Rightarrow dr = \frac{du}{-2r}$

$= \int -\frac{1}{2} u^{1/2} du$

$= -\frac{1}{3} u^{3/2} = -\frac{1}{3}(2-r^2)^{3/2}$

$$= \int_0^{2\pi} \left(-\frac{1}{3}(2-r^2)^{3/2} - \frac{1}{3} r^3\right) \Big|_{r=0}^{r=1} d\theta$$

$$= -\frac{1}{3} \int_0^{2\pi} \left((2-r^2)^{3/2} + r^3\right) \Big|_0^1 d\theta$$

$$= -\frac{1}{3} \int_0^{2\pi} \left[\left((2-1)^{3/2} + 1\right) - \left((2-0)^{3/2} + 0\right)\right] d\theta$$

$$= -\frac{1}{2} \int^{2\pi} \left(2 - 2\sqrt{2}\right) d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left(2-2\sqrt{2}\right)d\theta$$

$$= -\frac{2-2\sqrt{2}}{3}\int_0^{2\pi} d\theta$$

$$= -\frac{2-2\sqrt{2}}{3}\left[\theta\right]_0^{2\pi}$$

$$= -\frac{2-2\sqrt{2}}{3}\left(2\pi-0\right)$$

$$= \frac{4\pi}{3}\left(\sqrt{2}-1\right).$$

29–30 Evaluate the integral by changing to cylindrical coordinates.

30. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

From the given information, the region is

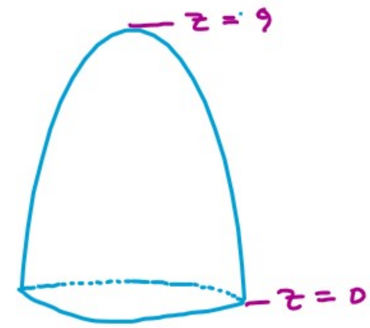
$$E = \left\{(x,y,z): -3\le x\le 3,\ 0\le y\le \sqrt{9-x^2},\ 0\le \underbrace{z\le 9-x^2-y^2}\right\}.$$

$z = 9-x^2-y^2 \Rightarrow z = 9-r^2$ and

$\Rightarrow x^2+y^2 = 9-z,\ z\ge 0$

From the plots (as shown), we obtain the region in cylindrical coordinates as

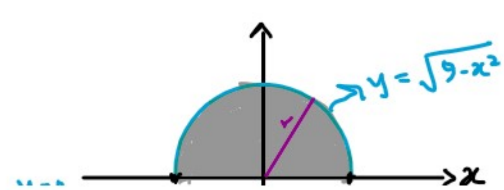
$$E = \left\{(r,\theta,z): 0\le\theta\le\pi,\ 0\le r\le 3,\ 0\le z\le 9-r^2\right\}.$$



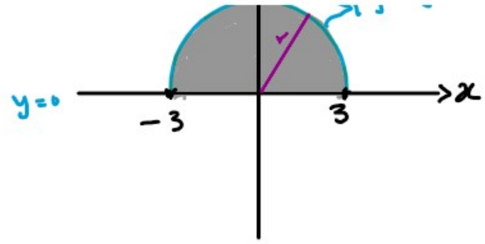
Thus,

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$

$$= \int_0^{\pi}\int_0^{3}\int_0^{9-r^2}\sqrt{r^2}\cdot r\,dz\,dr\,d\theta$$



$$= \int_0 \int_0 \int_0 \sqrt{r^2} \cdot r\,dz\,dr\,d\theta$$



$$= \int_0^{\pi} \int_0^{3} \int_0^{9-r^2} r^2\,dz\,dr\,d\theta$$

$$= \int_0^{\pi} \int_0^{3} r^2 z \Big|_{z=0}^{z=9-r^2} dr\,d\theta$$

$$= \int_0^{\pi} \int_0^{3} r^2 (9-r^2)\,dr\,d\theta$$

$$= \int_0^{\pi} \int_0^{3} (9r^2 - r^4)\,dr\,d\theta$$

$$= \int_0^{\pi} \left(3r^3 - \frac{1}{5}r^5\right)\Big|_0^3 d\theta$$

$$= \int_0^{\pi} \left[3(3^3) - \frac{1}{5}(3^5)\right] d\theta$$

$$= \left(81 - \frac{243}{5}\right) \int_0^{\pi} d\theta$$

$$= \frac{405-243}{5} (\theta)\Big|_0^{\pi}$$

$$= \frac{162}{5} (\pi)$$

$$= \frac{162\pi}{5}.$$