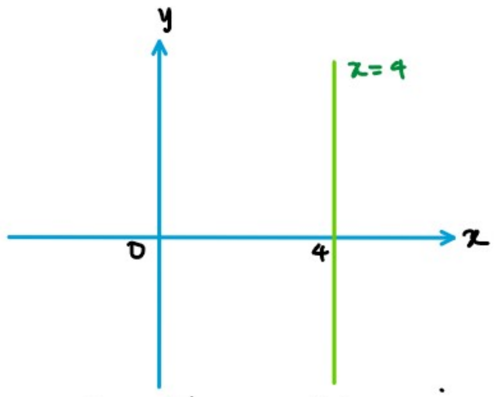
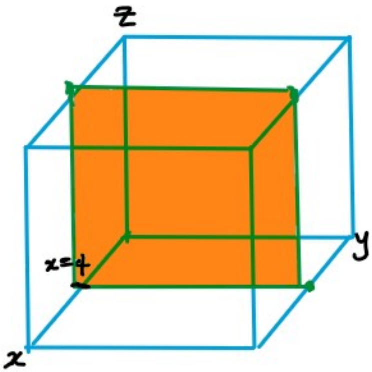
Sections 12.1: 5, 11, 13, 18, 44

5. What does the equation $x = 4$ represent in $\mathbb{R}^2$? What does it represent in $\mathbb{R}^3$? Illustrate with sketches.





$x=4$ is a line parallel to y-axis passing through the x-axis at 4 in $\mathbb{R}^2$.

In $\mathbb{R}^3$, $x=4$ is the set $\{(x,y,z): x=4\}$, a plane.

11. Determine whether the points lie on a straight line.

(a) $A(2, 4, 2)$, $B(3, 7, -2)$, $C(1, 3, 3)$

(b) $D(0, -5, 5)$, $E(1, -2, 4)$, $F(3, 4, 2)$

$$\underset{A}{\vdash}\!\!-\!\!\underset{B}{+}\!\!-\!\!\underset{C}{\dashv}$$

If A, B and C are on a straight line, as shown, then

$$|AC| = |AB| + |BC|$$

where $|AC|$ is the distance between A and C.

ie., Distance between the farthest points is the sum of the other distances.

(a) $|AB| = \sqrt{(2-3)^2 + (4-7)^2 + (2+2)^2} = \sqrt{1+9+16} = \sqrt{26}$

$|AC| = \sqrt{(2-1)^2 + (4-3)^2 + (2-3)^2} = \sqrt{1+1+1} = \sqrt{3}$

$|BC| = \sqrt{(3-1)^2 + (7-3)^2 + (-2-3)^2} = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$

Thus,

$$3\sqrt{5} \neq \sqrt{3} + \sqrt{26}$$

$\Rightarrow$ The points are not on a straight line.

(b) $|DE| = \sqrt{(0-1)^2 + (-5+2)^2 + (5-4)^2} = \sqrt{1+9+1} = \sqrt{11}$

$|DF| = \sqrt{(0-3)^2 + (-5-4)^2 + (5-2)^2} = \sqrt{9+81+9} = \sqrt{99} = 3\sqrt{11}$

$|EF| = \sqrt{(1-3)^2 + (-2-4)^2 + (4-2)^2} = \sqrt{4+36+4} = \sqrt{44} = 2\sqrt{11}$

Thus,

$$|DE| + |EF| = \sqrt{11} + 2\sqrt{11} = 3\sqrt{11} = |DF|$$

$\Rightarrow$ The three points lie on a straight line.

13. Find an equation of the sphere with center $(-3, 2, 5)$ and radius 4. What is the intersection of this sphere with the yz-plane?

Recall that

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

is the equation of a sphere with center (a, b, c) and radius r.

Thus,

$$(x+3)^2 + (y-2)^2 + (z-5)^2 = 4^2$$

$$\Rightarrow \quad (x+3)^2 + (y-2)^2 + (z-5)^2 = 16$$

is the required equation.

17–20 Show that the equation represents a sphere, and find its center and radius.

17. $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$

18. $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$

By completing the squares,

$$x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$$

$$\Rightarrow (x^2+8x) + (y^2-6y) + (z^2+2z) = -17$$

$$\Rightarrow (x^2+8x+16) + (y^2-6y+9) + (z^2+2z+1) = 16+9+1-17$$

$$\Rightarrow (x+4)^2 + (y-3)^2 + (z+1)^2 = 3^2$$

Hence, the equation represents a sphere with center $(-4, 3, -1)$ and radius 3.

44. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

Let $P = P(x, y, z)$. Then $|PA| = 2|PB|$.

$$\Rightarrow \sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = 2\sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2}$$

$$\Rightarrow (x+1)^2 + (y-5)^2 + (z-3)^2 = 4\left((x-6)^2 + (y-2)^2 + (z+2)^2\right)$$

$$\Rightarrow 4\left((x-6)^2 + (y-2)^2 + (z+2)^2\right) = (x+1)^2 + (y-5)^2 + (z-3)^2$$

$$\Rightarrow 4\left(x^2-12x+36 + y^2-4y+4 + z^2+4z+4\right) = x^2+2x+1 + y^2-10y+25 + z^2-6z+9$$

$$\Rightarrow 4(x^2-12x + y^2-4y + z^2+4z) + 176 = x^2+2x+y^2-10y+z^2-6z+35$$

$$\Rightarrow 4(x^2 - 12x + y^2 - 4y + z^2 + 4z) + 176 = x^2 + 2x + y^2 - 10y + z^2 - 6z + 35$$

$$\rightarrow (4x^2 - x^2 - 48x - 2x) + (4y^2 - y^2 - 16y + 10y) + (4z^2 - z^2 + 16z + 6z) = 35 - 176$$

$$\Rightarrow (3x^2 - 50x) + (3y^2 - 6y) + (3z^2 + 22z) = -141$$

$$\Rightarrow \left(x^2 - \frac{50}{3}x\right) + (y^2 - 2y) + \left(z^2 + \frac{22}{3}z\right) = -\frac{141}{3}$$

Now, complete the squares.

$$\Rightarrow \left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \left(\frac{25}{3}\right)^2 + 1^2 + \left(\frac{11}{3}\right)^2 - \frac{141}{3}$$

$$= \frac{625}{9} + 1 + \frac{121}{9} - \frac{141}{3}$$

$$= \frac{625 + 9 + 121 - 423}{9}$$

$$= \frac{332}{9}$$

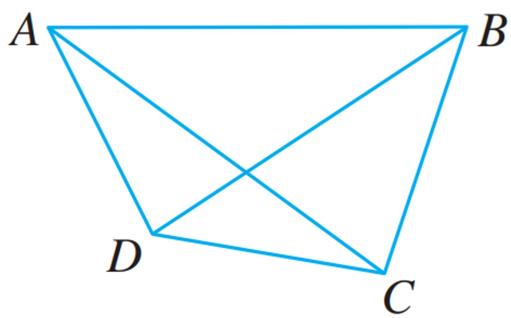
Hence, the set of all such points P lie on a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3}$.

Sections 12.2: 4, 17, 19, 24, 41

4. Write each combination of vectors as a single vector.

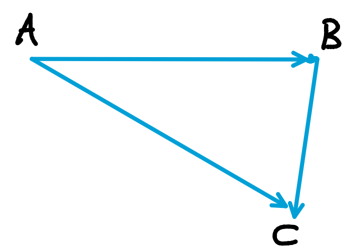
(a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{CD} + \overrightarrow{DB}$

(c) $\overrightarrow{DB} - \overrightarrow{AB}$ (d) $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AB}$

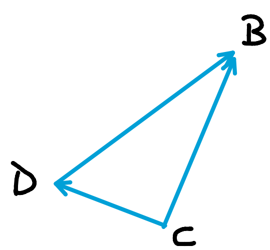


Using the triangle law in each case,

(a) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



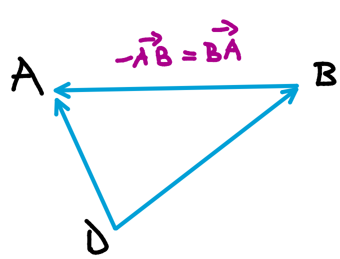
(b) $\overrightarrow{CD} + \overrightarrow{DB} = \overrightarrow{CB}$



(c) $\overrightarrow{DB} - \overrightarrow{AB} = \overrightarrow{DB} + (-\overrightarrow{AB})$

$= \overrightarrow{DB} + \overrightarrow{BA}$

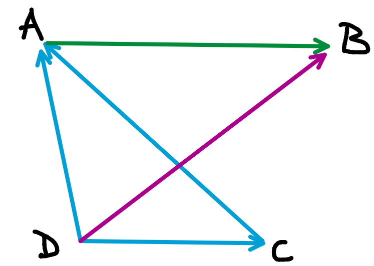
$= \overrightarrow{DA}$



(d) $\vec{DC} + \vec{CA} + \vec{AB} = (\vec{DC} + \vec{CA}) + (\vec{AB})$

$= \vec{DA} + \vec{AB}$

$= \vec{DB}$



15–18 Find the sum of the given vectors and illustrate geometrically.

15. $\langle -1, 4\rangle$, $\langle 6, -2\rangle$

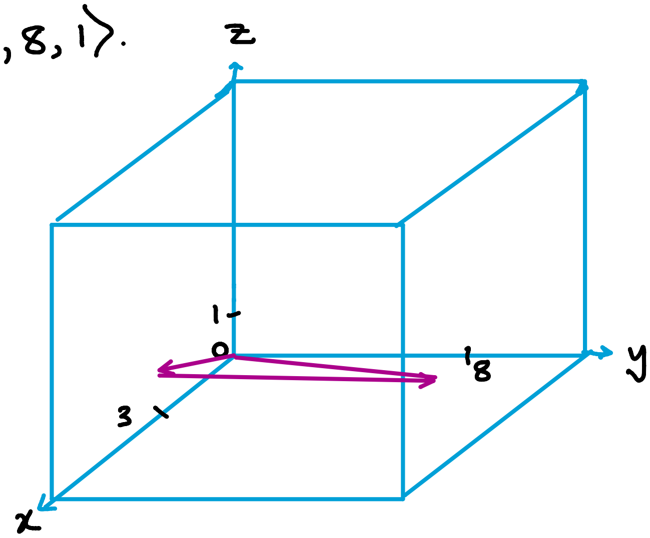
16. $\langle 3, -1\rangle$, $\langle -1, 5\rangle$

17. $\langle 3, 0, 1\rangle$, $\langle 0, 8, 0\rangle$

18. $\langle 1, 3, -2\rangle$, $\langle 0, 0, 6\rangle$

(17) $\langle 3, 0, 1\rangle + \langle 0, 8, 0\rangle = \langle 3+0, 0+8, 1+0\rangle$

$= \langle 3, 8, 1\rangle.$



19–22 Find $\mathbf{a} + \mathbf{b}$, $4\mathbf{a} + 2\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

19. $\mathbf{a} = \langle -3, 4\rangle$, $\mathbf{b} = \langle 9, -1\rangle$

$a + b = \langle -3, 4 \rangle + \langle 9, -1 \rangle = \langle -3+9, 4-1 \rangle = \langle 6, 3 \rangle.$

$4a + 2b = 4\langle -3, 4 \rangle + 2\langle 9, -1 \rangle = \langle -12, 16 \rangle + \langle 18, -2 \rangle = \langle 6, 14 \rangle$

$|a| = |\langle -3, 4 \rangle| = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

$a - b = \langle -3, 4 \rangle - \langle 9, -1 \rangle = \langle -3-9, 4+1 \rangle = \langle -12, 5 \rangle$

$|a-b| = |\langle -12, 5 \rangle| = \sqrt{(-12)^2 + 5^2} = \sqrt{144+25} = \sqrt{169} = 13$

23–25 Find a unit vector that has the same direction as the given vector.

23. $\langle 6, -2 \rangle$

24. $-5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

24 $|-5i + 3j - k| = \sqrt{(-5)^2 + 3^2 + (-1)^2} = \sqrt{25+9+1} = \sqrt{35}$

Hence, a unit vector with the same direction as the given vector is

$$\frac{1}{\sqrt{35}}(-5i + 3j - k)$$

41. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

$= 2(2) = 4$ is the slope of the tangent line

dx

$$\Rightarrow \quad \frac{dy}{dx}\bigg|_{x=2} = 2x\Big|_{x=2} = 2(2) = 4$$ is the slope of the tangent line to $y=x^2$ at $(2,4)$.

$(1,0) \quad (-1,-8)$

Thus, tangent line at $(2,4)$

$$y-4 = 4(x-2) \Rightarrow y = 4x-4$$

So $(1,0)$ and $(2,4)$ lie on the tangent line

So $\pm\langle 2-1, 4-0\rangle = \pm\langle 1,4\rangle$ are parallel vectors to the tangent line.

Since $$|\pm\langle 1,4\rangle| = \sqrt{1+16} = \sqrt{17},$$

it follows that the required unit vectors are $\pm\frac{1}{\sqrt{17}}(i+4j)$.