14.4 Tangent Planes and Linear Approximations

Definition Suppose f has continuous partial derviatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(a, b, c)$ is

$$z - c = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane namely,

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linerization** of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

Example 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

It is sometimes hard to use the definition of differentiable directly check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

Theorem If partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$

Definition For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the total differential, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Example 3

(a) If $z = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Homework

1, 6, 11, 25, 27, 29