14.6 Directional Derivatives and the Gradient Vector

1. The **directional derivative** of a two variable function f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

   if this limit exists.

2. The **directional derivative** of a three variable function f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

   if this limit exists.

Theorem 1. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

2. Similarly if f is a differentiable function of x, y, and z, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b, c\rangle$ and

$$D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

For a two variable functions, if the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = \langle\cos\theta, \sin\theta\rangle$ and the formula in Theorem 3 becomes

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Example 1 Find the directional derivative of $f(x, y) = x^3 - 3xy + 4y^2$ at the point $(1, 2)$ in the direction indicated by the angle $\theta = \pi/6$.

Example 2 Find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$

Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function, represented by ∇f, defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Similarly, if f is a function of three variables x, y, and z, then the **gradient** of f is

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

By the gradient vector of a differentiable function f we can rewrite the directional derivative as

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

For three variable function it is

$$D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$$

Example 3 For function $f(x,y,z) = x^2yz - xyz^3$, **(a)** Find the gradient of f, **(b)** Evaluate the gradient at the point $P(2,-1,1)$, **(c)** Find the rate of change of f at P in the direction of the vector $\mathbf{u} = \langle 0, 4/5, -3/5 \rangle$, **(d)** Express $D_{\mathbf{u}}f(x,y,z)$ by dot product of the gradient vector and the unit direction vector.

Homework 5, 6, 7, 10, 11, 13, 15