14.7 Maximum and Minimum Values

Let f be a two variable function;

1. If $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b) then f has a **local maximum** at (a,b). [This means that $f(x,y) \leq f(a,b)$ for all points (x,y) in some disk with center (a,b).] The number $f(a,b)$ is called a **local maximum value.**
2. If $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b), then f has a **local minimum** at (a,b). [This means that $f(x,y) \geq f(a,b)$ for all points (x,y) in some disk with center (a,b).] The number $f(a,b)$ is a **local minimum value**.

If the inequalities in the definition hold for all points (x,y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a,b).

Theorem If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

A point (a,b) is called a **critical point** (or *stationary point*) of f if $f_x(a,b)=0$ and $f_y(a,b)=0$, or if one of these partial derivatives does not exist.

Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [that is, (a,b) is a critical point of f]. Let

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

(c) If $D<0$, then $f(a,b)$ is not a local maximum or minimum.

Notes

1. In case (c) the point (a,b) is called a **saddle point** of f.
2. If $D=0$, the test gives no information: f could have a local maximum or local minimum at (a,b), or (a,b) could be a saddle point of f.

Example 1 Find the local maximum and minimum values and saddle point(s) of the function.

(a) $f(x,y) = xy - 2x - 2y - x^2 - y^2$

(b) $f(x,y) = y(e^x - 1)$

(c) $f(x,y) = 2 - x^4 + 2x^2 - y^2$

(d) $f(x,y) = (x^2+y^2)e^{-x}$

Example 2 Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0,1,1)$.

Example 3 Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.

Absolute Maximum and Minimum Values

Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum and minimum value; the smalest of these values is the absolute minimum value.

Example 4 Find the absolute maximum and minimum values of f on the set D.

(a) $f(x, y) = xy^2$, $D = \{(x, y) \mid \; x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$

(b) $f(x, y) = x^2 + y^2 + x^2y + 4$, $D = \{(x, y) \mid \; |x| \leq 1, |y| \leq 1\}$

Homework 7, 9, 12, 17, 32, 34, 35, 41, 45