14.8 Lagrange Multipliers

In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

Method of Lagrange Multipliers
To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ, called as **Lagrange multiplier**, such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a).
The largest of these values is the maximum value of f; the smallest is the minimum value of f.

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \quad g(x, y, z) = k.$$

This is a system of four equations in four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraints $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

This amounts to solving system of three equations in three unknowns:

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad g(x, y) = k.$$

Example 1 Find the extreme values of function $f(x, y) = xe^y$ on the circle $x^2 + y^2 = 2$.

Example 2 Find the extreme values of function $f(x, y, z) = e^{xyz}$ on the ellipsoid $2x^2+y^2+z^2 = 24$.

Example 3 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

Example 4 Use Lagrange multipliers to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the given constraint $x^4 + y^4 + z^4 = 1$.

Example 5 Find the extreme values of $f(x, y) = x^2+y^2+4x-4y$ on region $R = \{(x, y)|x^2+y^2 \leq 9\}$.

Homework 3, 5, 7, 9, 10, 20, 22