15.1 Double Integrals over Rectangles

Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

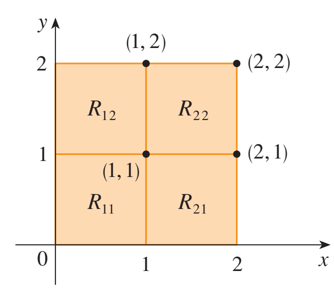
if this limit exists.

A function f is called **integrable** if the limit in the definition exists. The sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)$ is called a **double Riemann sum** and is used as an approximation to the value of the double integral.

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\ dA$$

Example 1 Estimate the volume of the solid that lies above the square $R = [0,2]\times[0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

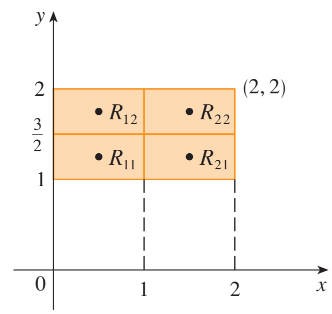


Midpoint Rule for Double Integrals

$$\iint_R f(x,y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example 2 Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\int\int_R (x - 3y^2)\ dA$, where $R = \{(x,y)|0 \leq x \leq 2, 1 \leq y \leq 2\}$.



Iterated Integrals

Suppose f is a function of two variables that is integrable on the ractangle $R = [a, b] \times [c, d]$, then

$$\int_R \int f(x, y)dydx = \int_a^b \left[\int_c^d f(x, y)dy \right] dx$$

The integral on the right side is called **iterated integral**.

Example 3 Evaluate the iterated integral $\int_{-3}^{3} \int_0^{\pi/2} (y + y^2 \cos x)dxdy$.

Fubini's Theorem If f is continuous on the rectangle $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 4 Evaluate $\int \int_R \frac{xy^2}{x^2 + 1} dA$, where $R = \{(x, y)|0 \leq x \leq 1, -3 \leq y \leq 3\}$.

$$\iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy \quad \text{where} \quad R = [a, b] \times [c, d]$$

The **Average Value** of a function of two variables defined on a rectangle R to be

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x, y)dA$$

where $A(R)$ is the area of R.

Example 5 Find the average value of $f = x^2y$ over the given rectangle whose vertices are $(-1,0), (-1,5), (1,5)$, and $(1,0)$

Homework 6, 8, 10, 18, 32, 34, 36, 48

Lecture Note 20, (Text book page 1001) 10/10/2018

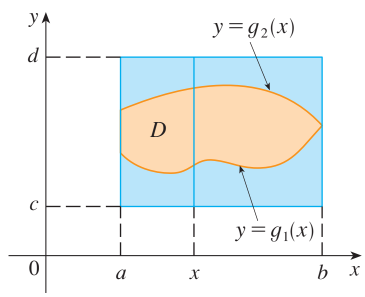
15.2 Double Integrals over General Regions

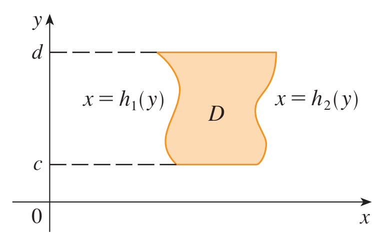
1. Suppose $g_1(x)$ and $g_2(x)$ are continuous functions on $[a, b]$ and the region R is defined by $R = \{(x, y) | g_1(x) \leq y \leq g_2(x); a \leq x \leq b\}$ (**type I**). Then

$$\iint_R f(x,y)dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy \right] dx$$

2. Suppose $h_1(y)$ and $h_2(y)$ are continuous functions on $[c, d]$ and the region R is defined by $R = \{(x, y) | h_1(x) \leq x \leq h_2(x); c \leq y \leq d\}$ (**type II**). Then

$$\iint_R f(x,y)dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx \right] dy$$





Example 1 Evaluate the double integral $\iint_D (x + 2y)dA$, D is the region bounded by $y = 2x^2$, $y = 1 + x^2$, $x \geq 0$.

Properties of Double Integrals

We assume that all of the following integrals exist.

1. $\iint\limits_D [f(x,y)+g(x,y)]dA = \iint\limits_D f(x,y)dA + \iint\limits_D g(x,y)dA$

2. $\iint\limits_D cf(x,y)dA = c\iint\limits_D f(x,y)dA$ where c is a constant.

3. If $f(x,y) \geq g(x,y)$ in D, then $\iint\limits_D f(x,y)dA \geq \iint\limits_D g(x,y)dA$

4. If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then $\iint\limits_D f(x,y)dA = \iint\limits_{D_1} f(x,y)dA + \iint\limits_{D_2} f(x,y)dA$

5. If we integrate constant function $f(x,y) = 1$ over a region D, we get the area of D: $\iint\limits_D 1dA = A(D)$

6. If $m \leq f(x,y) \leq M$ for all (x,y) in D, then $mA(D) \leq \iint\limits_D f(x,y)dA \leq MA(D)$.

Example 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the $xy-$ plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example 3 Evaluate the double integral $\iint\limits_D xy\ dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Example 4 Find the volume of the tetrahedron enclosed by the plane $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Example 5 Sketch the region of integration and change the order of integration. $\int_0^{\pi/2}\int_0^{\cos x} f(x,y)dydx$.

Example 6 Use Property 6 to estimate the integral $\iint\limits_D e^{\sin x \cos x}dA$, where D is the disk with center the origin and radius 2.

Homework 4, 8, 17, 21, 23, 25, 26, 28, 31, 36, 45, 49, 56, 60