15.3 Double Integrals in Polar Coordinates

Change to Polar Coordinates in a Dobule Integral

If $f(x, y)$ is continuous on a polar rectangle R given by $0 \le a \le r \le b, \quad \alpha \le \theta \le \beta,$ where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

The formula says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $rdrd\theta$.

Example 1 Sketch the region whose area is given by the integral $\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} rdrd\theta$ and evaluate the integral.

Example 2 Evaluate $\iint_R \frac{y^2}{y^2 + x^2} dA$ by changing to polar coordinate, where R is the region that lie between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$.

Example 3 Evaluate $\iint_D e^{-x^2-y^2} dA$ by changing to polar coordinate, where D is the region bounded by the semi-circle $x = \sqrt{4 - y^2}$ and the y-axis.

Example 4 Find the volume of the solid bounded by the plane $2x + y + z = 4$ and above disk $x^2 + y^2 \le 1$.

Example 5 Find the volume of the solid which is the intersection of the interior of the sphere $x^2 + y^2 + z^2 = 16$ and the outside of the cylinder $x^2 + y^2 = 4$.

If f is continuous on a polar region of the form $D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ then

$$\iint_D f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

Example 6 Use double integral to find the area enclosed by one loop of the four leaved rose $r = \cos 2\theta$.

Example 7 Use double integral to find the area enclosed by one loop of the three leaved rose $r = \cos 3\theta$.

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