15.6 Triple Integrals

We can define triple integrals for functions of three variables, as we defined single integrals for functions of one variable and double integrals for functions of two variables.

Let f be defined on a rectangular box $B = \{(x, y, z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$. Let's divide the interval $[a, b]$ into l subintervals of equal width Δx, divide $[c, d]$ into m subintervals of width Δy, and divide $[r, s]$ into n subintervals of width Δz. The planes through the end points of these subintervals parallel to the coordinate planes divde the box B into lmn sub-boxes, and each these boxes has volume $\Delta V = \Delta x \Delta y \Delta z$. Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

where the sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in B_{ijk}.

Definition The **triple integral** of f over the box B is

$$\iiint\limits_B f(x, y, z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

if this limit exists. The triple integral always exists if f is continuous.

If we choose the point (x_i, y_j, z_k), we get a simpler-looking expression for the triple integral:

$$\iiint\limits_B f(x, y, z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint\limits_B f(x, y, z)dV = \int_r^s \int_c^d \int_a^b f(x, y, z)dxdydz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z.

There are five other possible orders in which we can integrate, all of which give the same value. For example, if we integrate with respect to z, then x, and then y, we have

$$\iiint\limits_B f(x, y, z)dV = \int_c^d \int_a^b \int_r^s f(x, y, z)dzdxdy$$

Example 1 Evaluate the triple integral $\iiint\limits_B xyz^2 dV$, where B is the rectangular box given by $B = \{(x, y, z)|\ 0 \leq x \leq 1,\ -1 \leq y \leq 2,\ 0 \leq z \leq 3\}$

Now we define the **triple integral over a general bounded region** E in three dimensional space (a solid) by much the same procedure that we used for double integrals.

A solid region E is said to be **type 1** if it lies between the graphs of two continuous functions x and y, that is,

$$E = \{(x, y, z)|\ (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. So by this manner, after the first iteration, the triple integral becomes a double integral

$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz \right] dA$$

The rest of this is a solution of double integral.

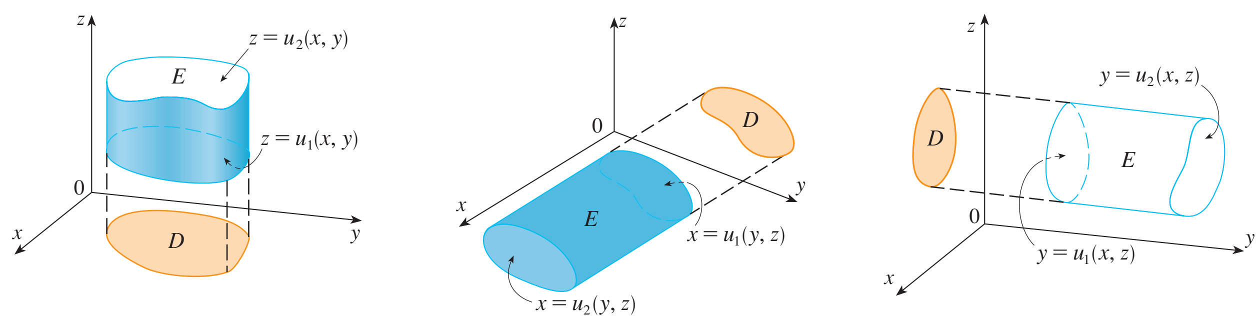
The triple integral becomes as follows if solid region E is defined as **type 2** of the form

$$E = \{(x, y, z)|\ (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\};\quad \iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx \right] dA$$

and the first iteration is based on x. Similarly for **type 3** the solid and triple integral becomes

$$E = \{(x, y, z)|\ (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\};\quad \iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy \right] dA$$

and for type 3, the first iteration is based y variable.



(a) A type 1 solid region (b) A type 2 region (c) A type 3 region

Example 2 Evaluate the triple integral $\displaystyle\iiint\limits_E y dV$ where

$$E = \{(x, y, z)|\ 0 \le x \le 3,\ 0 \le y \le x,\ x - y \le z \le x + y\}$$

Example 3 Evaluate the triple integral $\displaystyle\iiint\limits_E (x - y)dV$, where E is enclosed by the surfaces $z = x^2 - 1,\ z = 1 - x^2,\ y = 0,$ and $y = 2$.

Finding the volume of a solid by a triple integral

If $f(x, y, z) = 1$ for all points in E, then the triple integral represents the volume of E:

$$V(E) = \iiint_E dV$$

Example 4 Use a triple integral to find the volume of the solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

Example 5 Use a triple integral to find the volume of the solid enclosed by the cylinder $x^2+z^2 = 4$ and the planes $y = -1$, and $y + z = 4$.

Homework 4, 5, 7, 9, 13, 14, 19, 21