Lecture Note 23, (Text book page 1040) 10/23/2018

15.7 Triple Integrals in Cylindrical Coordinates

Cylindrical Coordinates

In three dimensions there is a coordinate system, called **cylindrical coordinates**, that is similar to polar coordinates.

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Evaluating Triple Integrals with Cylindrical Coordinates

Suppose that E is a **type 1** region whose projection D onto the xy-plane is conveniently described in polar coordinates. In particular suppose that f is continuous and

$$E = \{(x, y, z) | (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) | \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then the **formula for triple integration in cylindrical coordinates** becomes,

$$\iiint_E f(x, y, z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta$$

It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

Example 1 Evaluate $\iiint_E \sqrt{x^2 + y^2}\,dV$ where E is the region that lies inside cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

Example 2 Evaluate $\iiint_E x^2 dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

Example 3 Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.

Homework exercise 18, 22, 24, Example 3, Example 4.