# 15.8 Triple Integrals in Spherical Coordinates

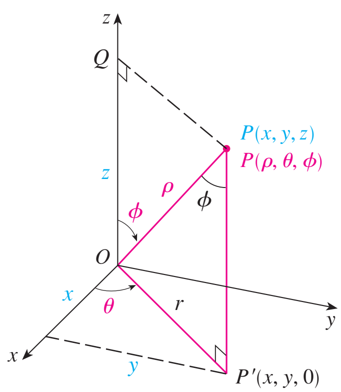
Another useful coordinate system in three dimensions is the **spherical coordinate system**. It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

> **Spherical Coordinates**
>
> The spherical coordinates $(\rho, \theta, \phi)$ of a point $P$ in space, where $\rho = |OP|$ is the distance from the origin to $P$, $\theta$ is the same angle as in cylindrical coordinates, and $\phi$ is the angle between the positive $z$-axis and the line segment $OP$. Note that
>
> $$\rho \geq 0 \quad \text{and} \quad 0 \leq \phi \leq \pi.$$
>
> The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

The relationship between rectangular and spherical coordinates can be seen as follows. From triangles $OPQ$ and $OPP'$ we have

$$z = \rho\cos\phi, \qquad r = \rho\sin\phi$$

(a)




(b) But $x = r\cos\theta$ and $y = r\cos\theta$, so to convert from spherical to rectangular coordinates, we use the equations

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad \text{and} \quad z = \rho\cos\phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

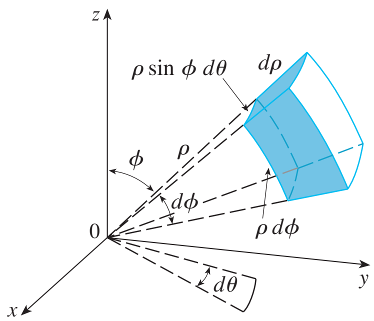
### Evaluating Triple integrals with spherical coordinates

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi)|\ a \leq \rho \leq b, \quad \alpha \leq \theta \leq \beta, \quad c \leq \phi \leq d\}$$

where $a \geq 0$ and $\beta - \alpha \leq 2\phi$, and $d - c \leq \phi$.

> **Formula for triple integration in spherical coordinates**
>
> $$\iiint_E f(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$
>
> where $E$ is a spherical wedge given by
>
> $$E = \{(\rho, \theta, \phi)|\ a \leq \rho \leq b, \quad \alpha \leq \theta \leq \beta, \quad c \leq \phi \leq d\}$$

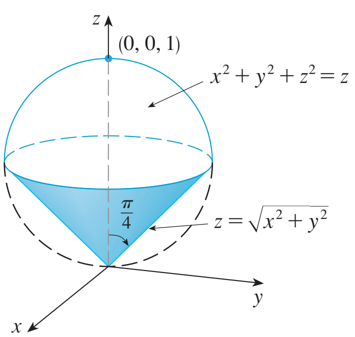



**Example 1** The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

**Example 2** The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

**Example 3** Evaluate $\iiint\limits_E e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$, where $B$ is the unit ball: $B = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$

**Example 4** Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.




**Homework** 4, 5, 7, 11, 13, 15, 19, 21