Chapter 16 Vector Complex

16.1 Vector Fields

Definition (1). Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $F(x, y)$.

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

(2). Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $F(x, y, z)$.

We can express a vector field in three dimensional space in terms of its component functions P, Q, and R as

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

Example 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Described $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$.

Example 2 Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\mathbf{k}$.

Gradient Fields

If f is a scalar function of two variables, then its gradient ∇f is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**.
Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ is given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

Example 3 Find the gradient vector field of $f(x.y, z) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

Homework 1, 11, 13.

16.2 Line Integrals

Outcome 1: Evaluate the line integral of a function along a piecewise smooth curve with respect to arc length.

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C, and such integrals are called *line integrals.*

We start with a plane curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b \tag{1}$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, and we assume that C is a smooth curve. (This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$).

Definition (1) If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

In previous classes, section 10.2, the length of C should have been shown as

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

A similar type of argument can be used to show that if f is a continuous function, then the limit in the definition always exists and the following formula can be used to evaluate the line integral:

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

If f is a positive function i.e. $f(x, y) \geq 0$, $\int_C f(x, y)ds$ represents the area of one side of the "fence" or "curtain", whose base is C and whose height above the point (x, y) is $f(x, y)$.

Example 1 Evaluate the line integral $\int_C xy^4 ds$ where C is the right half of the circle $x^2 + y^2 = 16$.

If C is a **piecewise-smooth curve**; that is C is a union of a finite number of smooth curves $C_1, C_2, \cdots, C_n$, then

$$\int_C f(x, y)ds = \int_{C_1} f(x, y)ds + \int_{C_2} f(x, y)ds + \cdots + \int_{C_n} f(x, y)ds$$

Example 2 Evaluate $\int_C 2xds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

Outcome 2: Find the mass and center of mass of a thin wire given its shape and linear density.

Any physical interpretation of a line integral $\int_C f(x,y)ds$ depends on the physical interpretation of the function f. Suppose that $\rho(x,y)$ represents the linear density at a point (x,y) of a thin wire shaped like a curve C. Then the mass of the part of the wire from P_{i-1} to P_i is approximately $\rho(x_i^*, y_i^*)\Delta s_i$ and so the total mass of the wire is approximately $\sum \rho(x_i^*, y_i^*)\Delta s_i$. By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\Delta s_i = \int_C \rho(x,y)ds$$

The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)ds, \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)ds$$

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition ($\star$). They are called the **line integrals of f along C with respect to x and y**, They can be evaluated by expressing everything in terms of t: $x = x(t), y = y(t), dx = x^{'}(t)dt$, $dy = y^{'}(t)dt$,

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))x^{'}(t)dt \qquad \int_C f(x,y)dy = \int_a^b f(x(t), y(t)y^{'}(t)dt$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given. In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \qquad 0 \le t \le 1$$

Example 3 Evaluate the line integral $\int_C x^2dx + y^2dy$, C consists of the arc of the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$.

Line Integrals in Space

Outcome 3: Evaluate the line integral of a function along piecewise smooth curve with respect to x, y, or z.

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$. The definition line integral in space can be given by a similar method of two dimensional space. We evaluate the line integral in space as following,

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

- The line integrals in 2D and 3D can be written in the more compact notation $\displaystyle\int_a^b f(\mathbf{r}(t))|\mathbf{r}^{'}(t)|dt$.
- For the special case $f(x,y,z) = 1$, we get $\displaystyle\int_C ds = \int_a^b |\mathbf{r}^{'}(t)|dt = L$ where L is the length of the curve C.
- Line integrals along C with respect to x, y, and z can also be defined as follow;
  $\displaystyle\int_C f(x,y,z)dx = \int_a^b f(x(t),y(t),z(t))x^{'}(t)dt$
  $\displaystyle\int_C f(x,y,z)dy = \int_a^b f(x(t),y(t),z(t))y^{'}(t)dt$
  $\displaystyle\int_C f(x,y,z)dz = \int_a^b f(x(t),y(t),z(t))z^{'}(t)dt$
- Therefore, as with line integrals in the plane, we evaluate integrals of the form

  $$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz \cdots\cdots (2)$$

  by expressing everthing (x, y, z, dx, dy, dz) in terms of the parameter t.

Example 4 Evaluate the line integral, $\int_C x^2y\ ds$, where the curve $C:\ x = \cos t,\ y = \sin t,\ z = t$, $0 \le t \le \pi/2$

Example 5 Evaluate the line integral, $\int_C xye^{yz}\ dy$, where the curve $C:\ x = t,\ y = t^2,\ z = t^3$, $0 \le t \le 1$

Line Integrals of Vector Fields

Outcome 4: Evaluate the line integral of a vector field along a piecewise smooth curve.

> **Definition** (2) Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **the line integral of F along** C is
>
> $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}^{'}(t)dt = \int_C \mathbf{F}\cdot\mathbf{T}ds$$

When using Definition (2), bear in mind that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for the vector field $\mathbf{F}(x(t), y(t), z(t))$, so we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply puting $x = x(t), y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$. Notice also that we can formally write $d\mathbf{r} = \mathbf{r}^{'}(t)dt$.

Example 6 Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x,y,z) = (x+y^2)\mathbf{i} + xz\mathbf{j} + (y+z)\mathbf{k}, \quad \mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} - 2t\mathbf{k}, \quad 0 \le t \le 2.$$

Suppose the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. We use Definition (2) to compute its line integral along C.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}^{'}(t)dt \\
&= \int_a^b (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x^{'}(t)\mathbf{i} + y^{'}(t)\mathbf{j} + z^{'}(t)\mathbf{k}) \\
&= \int_a^b \Big[P(x(t), y(t), z(t))x^{'}(t) + Q(x(t), y(t), z(t))y^{'}(t) + R(x(t), y(t), z(t))z^{'}(t) \Big] dt
\end{aligned}$$

But this last integral is precisely the line integral in (2). Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz \qquad \text{where} \qquad \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

The line integral of $\mathbf{F}$ along the curve C represents the **work done** by the tangential component of the force $\mathbf{F}$ along C.

Homework 2, 7, 10, 15, 21, 22, 35