16.5 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism.

Curl

Definition If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

An easy way to remember this formula is that curl $\mathbf{F} = \nabla \times \mathbf{F}$ where the vector differential operator ∇ ("del") is defined as $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$.

Example 1 If $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$, find curl F.

Example 2-(Theorem) If f is a function of three variables that has continuous second order derivatives, then curl $(\nabla f) = \mathbf{0}$.

Since a conservative vector field is one for which $F = \nabla f$, example 2-(Theorem) can be re phrased as follows:

If F is conservative, then curl $F = 0$. (This gives us a way of verifying that a vector field is not conservative.)

Example 3 Show that $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ is not conservative.

Theorem If $\mathbf{F}$ is a vector field defined on all $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Example 4 Show that $\mathbf{F}(x, y, z) = \mathbf{i} + \sin z\mathbf{j} + y\cos z\mathbf{k}$ is a conservative vector field.

Divergence

Definition If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x, \partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$

Observe that curl $\mathbf{F}$ is a vector field but div $\mathbf{F}$ is a scalar field.

Example 5 If $\mathbf{F}(x, y, z) = \langle xz,\ xyz,\ -y^2\rangle$, find div $\mathbf{F}$.

Theorem ($\star$) If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Example 6 By considering Theorem ($\star$) explain whether there is a vector G or not on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$?

Homework 1, 3, 12, 13, 17, 19