Gaussian Quadrature

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Among the many quadrature rules available for approximating

$$\int_a^b f(x)\,dx,$$

Gaussian Quadrature rule is one of the most powerful in terms of accuracy and ease of computation.

Here, we illustrate how and why this rule works.

Suppose we want to approximate $\int_a^b f(x)\,dx$ via a polynomial of degree three (for instance). Then this warrants finding a_0, a_1, a_2 and a_3 such that

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

and

$$\int_a^b f(x)\,dx \approx \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

Gaussian Quadrature rule recasts this problem into a two-points function evaluations through

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2) \quad \text{———} \; ①$$

and using the u-sub

$$u = \frac{b-a}{2}x + \frac{b+a}{2}$$

(gotten by finding the parametric line between $(-1, a)$ and $(1, b)$)

to get that

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

$$\approx \frac{b-a}{2}\left[c_1 f\left(\frac{b-a}{2}x_1 + \frac{b+a}{2}\right) + c_2 f\left(\frac{b-a}{2}x_2 + \frac{b+a}{2}\right)\right] \quad \text{—— (2)}$$

Equation (1) (and or (2)) is called two points function evaluations because c_1, c_2, x_1 and x_2 are precomputed and notice that there are four of them (c_1, c_2, x_1, x_2) corresponding to the four coefficients (a_0, a_1, a_2, a_3) of the approximating polynomial.

COMPUTING c_1, c_2, x_1, x_2 INSTEAD OF a_0, a_1, a_2, a_3

The power of Gaussian quadrature method lies in the fact that c_1, c_2, x_1, x_2 are computed once and they apply to all functions unlike finding a_0, a_1, a_2, a_3 which change once the function $f(x)$ changes.

Suppose the integral $\int_{-1}^{1} f(x)\,dx$ is exact up to a polynomial of degree 3.

i.e.,

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Then, we are looking to find c_1, c_2, x_1, x_2 such that

$$\int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\,dx$$

$$= c_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

$$= a_0(c_1 + c_2) + a_1(c_1 x_1 + c_2 x_2) + a_2(c_1 x_1^2 + c_2 x_2^2) + a_3(c_1 x_1^3 + c_2 x_2^3)$$

$$\Rightarrow \left(a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \frac{a_3}{4} x^4\right)\Big|^{1} = \int^{1} (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\,dx$$

$$\Rightarrow \left(a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4\right)\Bigg|_{-1}^{1} = \int_{-1}^{1}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= a_0(c_1+c_2) + a_1(c_1x_1+c_2x_2) + a_2(c_1x_1^2+c_2x_2^2) + a_3(c_1x_1^3+c_2x_2^3)$$

$$\Rightarrow a_0(1+1) + \frac{a_1}{2}(1-1) + \frac{a_2}{3}(1+1) + \frac{a_3}{4}(1-1) = a_0(c_1+c_2) + a_1(c_1x_1+c_2x_2) + a_2(c_1x_1^2+c_2x_2^2) + a_3(c_1x_1^3+c_2x_2^3)$$

$$\Rightarrow 2a_0 + \frac{2}{3}a_2 = a_0(c_1+c_2) + a_1(c_1x_1+c_2x_2) + a_2(c_1x_1^2+c_2x_2^2) + a_3(c_1x_1^3+c_2x_2^3)$$

Comparing LHS and RHS term by term

$$\Rightarrow \quad c_1 + c_2 = 2 \quad \text{—— (i)}$$

$$c_1x_1 + c_2x_2 = 0 \quad \text{—— (ii)}$$

$$c_1x_1^2 + c_2x_2^2 = \frac{2}{3} \quad \text{—— (iii)}$$

$$c_1x_1^3 + c_2x_2^3 = 0 \quad \text{—— (iv)}$$

$$\text{(i)} \times \left(-\tfrac{1}{2}\right): \quad -\frac{1}{2}c_1 - \frac{1}{2}c_2 = -1$$

$$\text{(iii)} \times \frac{3}{2}: \quad \frac{3}{2}c_1x_1^2 + \frac{3}{2}c_2x_2^2 = 1$$

$$\text{Adding:} \quad \frac{3}{2}c_1x_1^2 - \frac{1}{2}c_1 + \frac{3}{2}c_2x_2^2 - \frac{1}{2}c_2 = 0$$

$$\Rightarrow \frac{c_1}{2}(3x_1^2 - 1) + \frac{c_2}{2}(3x_2^2 - 1) = 0$$

$$\Rightarrow c_1 P_2(x_1) + c_2 P_2(x_2) = 0, \quad P_2(x) = \frac{1}{2}(3x^2-1). \quad \text{—— (v)}$$

Clearly, if $P_2(x) = 0$, ie, $\frac{1}{2}(3x^2-1) = 0$

$$\Rightarrow x = \pm\frac{1}{\sqrt{3}},$$

then (v) is satisfied at these points: $x_1 = -\frac{1}{\sqrt{3}}$ and $x_2 = \frac{1}{\sqrt{3}}$.

Plugging $x_1 = -\frac{1}{\sqrt{3}}$ and $x_2 = \frac{1}{\sqrt{3}}$ into (ii), (or (iv))

$$-\frac{1}{\sqrt{3}}c_1 + \frac{1}{\sqrt{3}}c_2 = 0 \qquad \text{OR} \qquad \left(-\frac{1}{\sqrt{3}}\right)^3 c_1 + \left(\frac{1}{\sqrt{3}}\right)^3 c_2 = 0$$

$$\Rightarrow c_1 = c_2. \qquad\qquad \Rightarrow c_1 = c_2.$$

So from (i), we get:

$$c_1 + c_2 = 2 \Rightarrow 2c_1 = 2 \Rightarrow c_1 = 1 = c_2.$$

Hence,

$$c_1 = 1 = c_2 \text{ and } x_1 = -\frac{1}{\sqrt{3}}, \; x_2 = \frac{1}{\sqrt{3}}.$$

Thus,

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

is exact up to degree 3 polynomial.

Furthermore,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[c_1 f\left(\frac{b-a}{2}x_1 + \frac{b+a}{2}\right) + c_2 f\left(\frac{b-a}{2}x_2 + \frac{b+a}{2}\right)\right]$$

$$= \frac{b-a}{2}\left[f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)\right]$$

is exact up to a degree 3 polynomial approximation of f and the integral is, in fact, converted to just two-points evaluations of f.

NB:

* $c_1 = 1 = c_2$ and $x_1 = -\frac{1}{\sqrt{3}}$, $x = \frac{1}{\sqrt{3}}$ satisfy equations (i) – (iv).

* Moreover, similarly performing

$$\text{(ii)} \times \left(-\tfrac{1}{2}\right): \quad -\tfrac{1}{2} c_1 x_1 - \tfrac{1}{2} c_2 x_2 = 0$$

$$\text{(iv)} \times \left(\tfrac{3}{2}\right): \quad \underline{\tfrac{3}{2} c_1 x_1^3 + \tfrac{3}{2} c_2 x_2^3 = 0}$$

and adding: $\frac{3}{2} c_1 x_1^3 - \frac{1}{2} c_1 x_1 + \frac{3}{2} c_2 x_2^3 - \frac{1}{2} c_2 x_2 = 0$

$$\Rightarrow \quad \frac{c_1 x_1}{2}\left(3x_1^2 - 1\right) + \frac{c_2 x_2}{2}\left(3x_2^2 - 1\right) = 0$$

$$\Rightarrow \quad c_1 x_1 P_2(x_1) + c_2 x_2 P_2(x_2) = 0, \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

Again, we see that the roots of P_2 solves this equation.

* c_1 and c_2 are called Gauss weights while x_1 and x_2 are called Gauss points.

* Since we used two Gauss points for obtaining the integral that is exact up to a degree three polynomial approximation, this method is called two-points Gaussian quadrature rule

* We were required to find 4 points c_1, c_2, x_1, x_2 for a two-points Gaussian quadrature rule and the corresponding 4 coefficients of a degree 3 polynomial played crucial role in obtaining 4 equations in c_1, c_2, x_1, x_2 which had a unique solution.

* In general, for n-points Gaussian quadrature, we are required to find $c_1, c_2, \cdots, c_n$; $x_1, x_2, \cdots, x_n$ and for this, we need an approximating polynomial with $2n$ coefficients in order to get $2n$ equations. Appropriate to this is a polynomial of degree $2n-1$. So n-points Gaussian quadrature rule approximates $\int_a^b f(x)dx$ exactly up to a polynomial of degree $2n-1$ (Powerful!)

* The coefficients a_0, a_1, a_2, a_3 of the approximating polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

are not present in the final approximate solution

$$\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) = f\left(\tfrac{-1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right).$$

In general, this is always the case since the coefficients are only used in deriving the $2n$ equations.
The power of Gaussian quadrature rule comes from this fact since the independence of the integral on the approximating polynomial coefficients guarantees that the weights $c_1, c_2, \cdots, c_n$ and the Gauss points $x_1, x_2, \cdots, x_n$ once computed for each n, can be used to approximate the integral of any function exactly up to a polynomial of degree $2n-1$.

* The polynomial P_2 is the Legendre Polynomial that solves the Legendre differential equation

$$(x^2-1)y'' - 2xy' + k(k+1)y = 0$$

for $k=2$.

In general, P_n given by the Rodrigue's formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

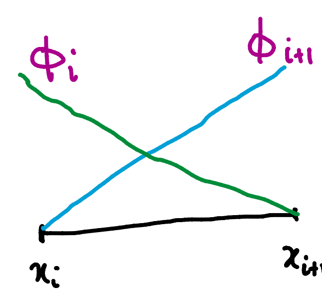
solves the Legendre ODE for $k=n$.

* Solution to the system of equations resulting from finding Gauss weights and Gauss points/nodes always depends on finding the root(s) of at least one of these polynomials.

GAUSS NODES ON SHAPE FUNCTIONS

Over the element $[x_i, x_{i+1}]$, a linear shape function is given as

$$\phi_i(x) = \frac{x_{i+1}-x}{x_{i+1}-x_i}$$



$$\phi_{i+1}(x) = \frac{x-x_i}{x_{i+1}-x_i}.$$

To approximate the integral $\int_{x_i}^{x_{i+1}} f(x)\phi_i(x)\,dx$ and $\int_{x_i}^{x_{i+1}} f(x)\phi_{i+1}(x)\,dx$

To approximate the integral $\int_{x_i} f(x)\,\phi_i(x)\,dx$ and $\int_{x_i} f(x)\,\phi_{i+1}(x)\,dx$

using Gaussian quadrature rule, we use the transformation

$$x = \frac{x_{i+1}-x_i}{2}r + \frac{x_{i+1}+x_i}{2}, \quad r \in [-1,1]$$

to convert them to integrals of the form

$$\int_{-1}^{1} f(x(r))\,\phi_i(x(r))\,dr \quad \text{and} \quad \int_{-1}^{1} f(x(r))\,\phi_{i+1}(x(r))\,dr$$

since Gaussian nodes/points and weights are derived on $[-1,1]$. To simplify the number of evaluations required, we notice that

$$\phi_i(x) = \frac{x_{i+1}-x}{x_{i+1}-x_i}$$

$$= \frac{x_{i+1} - \left(\frac{x_{i+1}-x_i}{2}r + \frac{x_{i+1}+x_i}{2}\right)}{x_{i+1}-x_i}$$

$$= \frac{2x_{i+1} - (x_{i+1}-x_i)r - x_{i+1} - x_i}{2(x_{i+1}-x_i)}$$

$$= \frac{x_{i+1}-x_i - (x_{i+1}-x_i)r}{2(x_{i+1}-x_i)}$$

$$= \frac{x_{i+1}-x_i}{2(x_{i+1}-x_i)} - \frac{(x_{i+1}-x_i)r}{2(x_{i+1}-x_i)}$$

$$= \frac{[illegible]}{2(x_{i+1}-x_i)} - \frac{[illegible]}{2(x_{i+1}-x_i)}$$

$$= \frac{1}{2} - \frac{1}{2}\gamma$$

So instead of finding $x_j = x(\gamma_j)$ for each Gaussian node γ_j and then computing $\phi_i(x_j)$ at each Gaussian node, we now have

$$\phi_i(x_j) = \phi_i(x(\gamma_j))$$

$$= \frac{1}{2} - \frac{1}{2}\gamma_j .$$

Similarly,

$$\phi_{i+1}(x) = \frac{x - x_i}{x_{i+1}-x_i}$$

$$= \frac{\left(\frac{x_{i+1}-x_i}{2}\gamma + \frac{x_{i+1}+x_i}{2}\right) - x_i}{x_{i+1}-x_i}$$

$$= \frac{(x_{i+1}-x_i)\gamma + x_{i+1}+x_i - 2x_i}{2(x_{i+1}-x_i)}$$

$$= \frac{(x_{i+1}-x_i)\gamma + x_{i+1}-x_i}{2(x_{i+1}-x_i)}$$

$$= \frac{(x_{i+1}-x_i)\gamma}{2(x_{i+1}-x_i)} + \frac{x_{i+1}-x_i}{2(x_{i+1}-x_i)}$$

$$= \frac{(x_{i+1} - x_i)\,[\text{illegible}]}{2(x_{i+1} - x_i)} + \frac{[\text{illegible}]}{2(x_{i+1} - x_i)}$$

$$= \frac{1}{2} r + \frac{1}{2}$$

So at each Gaussian node j,

$$\phi_{i+1}(x_j) = \phi_{i+1}(x(r_j))$$

$$= \frac{1}{2} + \frac{1}{2} r_j$$