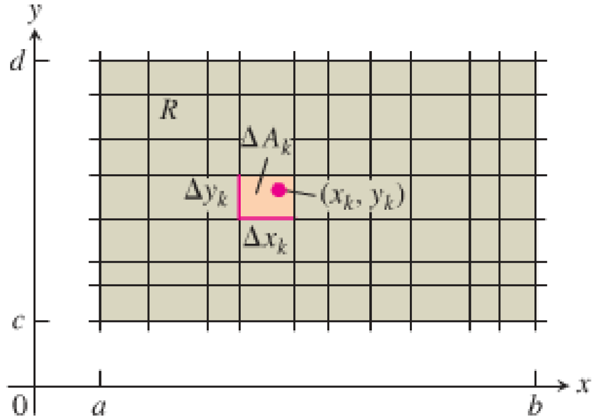
Double Integrals

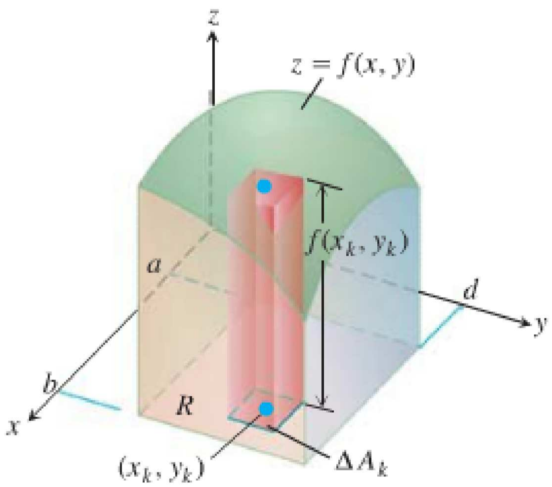
Consider a function $f(x, y)$ defined on a rectangular region $R: \ a \leq x \leq b, \ c \leq y \leq d$ partitioned into small rectangles A_k:



The area of a small rectangle with sides Δx_k and Δy_k is

$$\Delta A_k = \Delta x_k \, \Delta y_k \, .$$

Choose a point (x_k, y_k) in the (suitably numbered) kth rectangle with function value $f(x_k, y_k)$. We can consider $z = f(x, y)$ as defining the height z at the point (x, y). The product $f(x_k, y_k)\,\Delta A_k$ is then the *volume of a solid* with base area ΔA_k and height $f(x_k, y_k)$ (for which we assume that $f(x_k, y_k) > 0$):



The **Riemann sum** S_n of these solids over R is

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k\,.$$

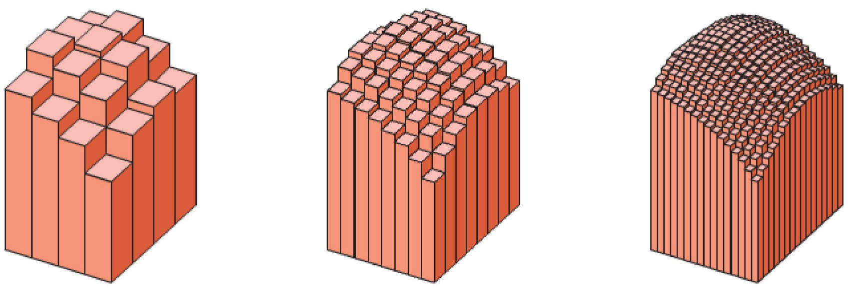
Now consider what happens as $\Delta A_k \to 0$ (as $n \to \infty$), i.e., we refine the partitioning. When the limit of these sums exists the function f is said to be **integrable** and the limit is called the **double integral** of f over R, written as

$$\int_R\!\!\int f(x, y)\,\mathrm{d}A \qquad \text{or} \qquad \int_R\!\!\int f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

The volume of the portion of the solid directly above the base ΔA_k is $f(x_k, y_k)\,\Delta A_k$. Hence the total volume above the region R is

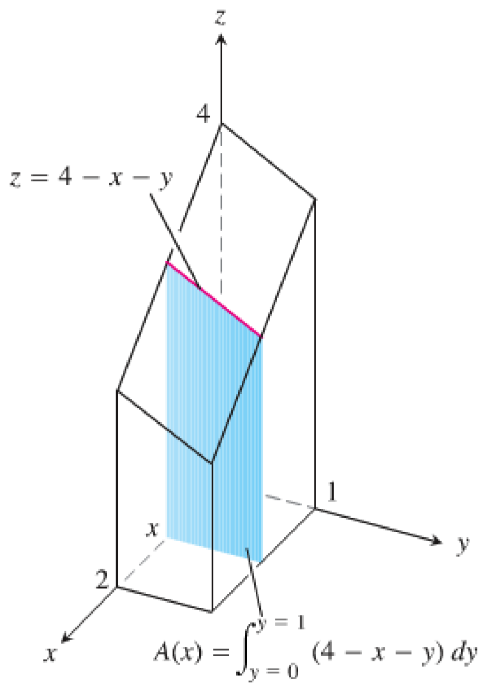
$$\text{Volume} \;=\; \lim_{n\to\infty} S_n = \int_R\!\!\int f(x, y)\,\mathrm{d}A$$

where $\Delta A_k \to 0$ as $n \to \infty$. The following figure shows how the Riemann sum approximations of the volume become more accurate as the number n of boxes increases:



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Consider the calculation of the volume under the plane $z = 4 - x - y$ over the rectangular region $R:\ 0 \le x \le 2$ and $0 \le y \le 1$ in the x-y plane.
First consider a slice perpendicular to the x-axis:



The volume under the plane is

$$\int_{x=0}^{x=2} A(x)\,\mathrm{d}x$$

where $A(x)$ is the cross-sectional area at x. For each value of x we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\,\mathrm{d}y$$

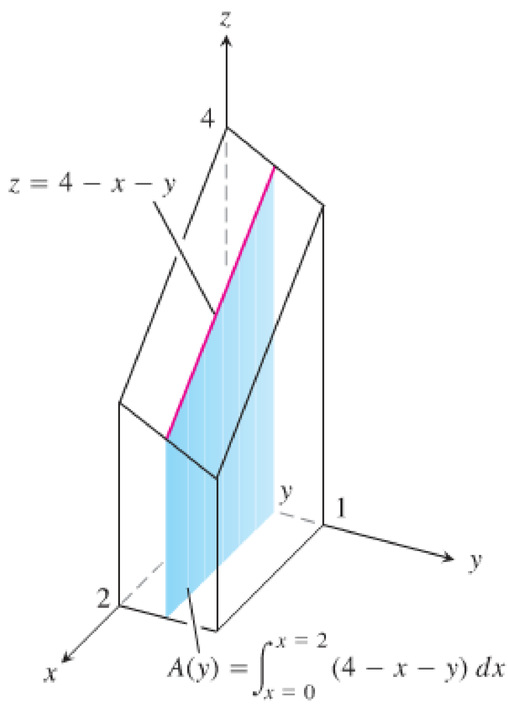
which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x.
In calculating $A(x)$, x is held fixed and the integration takes place with respect to y.
Combining the above two equations we have

$$\begin{aligned}
\text{Volume} &= \int_{x=0}^{x=2} A(x)\,\mathrm{d}x \\
&= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} \mathrm{d}x = \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) \mathrm{d}x \\
&= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = \left(\frac{7}{2} \cdot 2 - \frac{2^2}{2} \right) - (0 - 0) = 5\,.
\end{aligned}$$

We can write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

This is an **iterated** or **repeated integral**. The expression states that we can get the volume under the plane by (i) integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$, holding x fixed, and then (ii) integrating the resulting expression in x from $x = 0$ to $x = 2$. In other words, *first do the* dy *integral and then do the* dx *integral.*
Now consider the plane perpendicular to the y-axis:



We have

$$A(y) = \int_{x=0}^{x=2} (4 - x - y)\,\mathrm{d}x = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2} = 6 - 2y\,.$$

The volume is then

$$\text{Volume} = \int_{y=0}^{y=1} A(y)\,\mathrm{d}y = \int_{y=0}^{y=1} (6 - 2y)\,\mathrm{d}y = \left[6y - y^2\right]_0^1 = 5$$

as before.
This illustrates

THEOREM 1 Fubini's Theorem (First Form)
If $f(x, y)$ is continuous throughout the rectangular region $R\colon a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx.$$

Example:
Calculate the volume V under $z = f(x,y) = x^2y$ over the rectangle R defined by $1 \le x \le 2$, $-3 \le y \le 4$.

$$\begin{aligned} V &= \int\!\!\int_R x^2y\,\mathrm{d}A = \int_{x=1}^{x=2}\left(\int_{y=-3}^{y=4} x^2y\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{x=1}^{x=2}\left[\frac{x^2y^2}{2}\right]_{y=-3}^{y=4}\mathrm{d}x = \int_{x=1}^{x=2}\frac{7x^2}{2}\,\mathrm{d}x = \left[\frac{7x^3}{6}\right]_{x=1}^{x=2} = \frac{49}{6}\,. \end{aligned}$$

Changing the order gives the same result:

$$\begin{aligned} V &= \int\!\!\int_R x^2y\,\mathrm{d}A = \int_{y=-3}^{y=4}\left(\int_{x=1}^{x=2} x^2y\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{y=-3}^{y=4}\left[\frac{x^3y}{3}\right]_{x=1}^{x=2}\mathrm{d}y = \int_{y=-3}^{y=4}\frac{7y}{3}\,\mathrm{d}y = \left[\frac{7y^2}{6}\right]_{y=-3}^{y=4} = \frac{49}{6}\,. \end{aligned}$$

In this example we could have separated the integrand into its x and y parts:

$$V = \int_{x=1}^{x=2}\left(\int_{y=-3}^{y=4} x^2y\,\mathrm{d}y\right)\mathrm{d}x = \left(\int_{x=1}^{x=2} x^2\,\mathrm{d}x\right)\left(\int_{y=-3}^{y=4} y\,\mathrm{d}y\right) = \frac{7}{3}\cdot\frac{7}{2} = \frac{49}{6}\,.$$

More generally, if $f(x,y) = g(x)\,h(y)$, (i.e. the function is **separable**) *and* the region is **rectangular** then

$$\begin{aligned} \int\!\!\int_R g(x)\,h(y)\,\mathrm{d}A &= \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} g(x)\,h(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \left(\int_{x=a}^{x=b} g(x)\,\mathrm{d}x\right)\left(\int_{y=c}^{y=d} h(y)\,\mathrm{d}y\right)\,. \end{aligned}$$

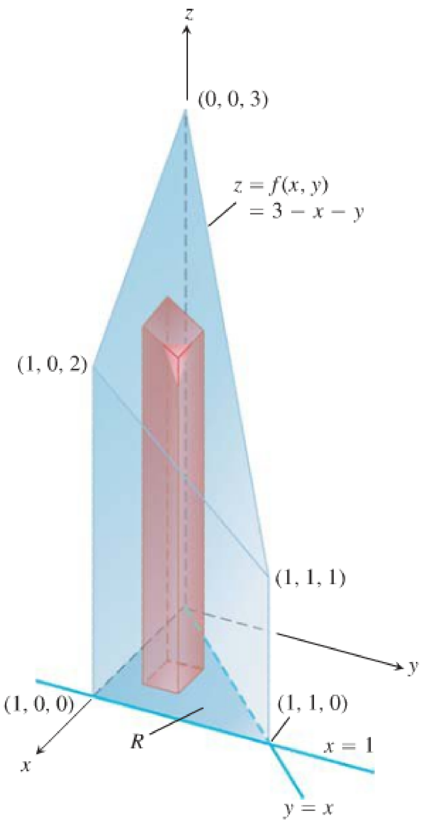
Now consider the case where the region R is *not rectangular*:[1]

THEOREM 2 Fubini's Theorem (Stronger Form)
Let $f(x, y)$ be continuous on a region R.

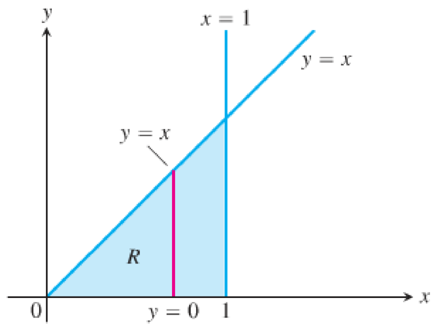
1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

[1] see Thomas' Calculus, beginning of Section 15.2 for details underlying this theorem

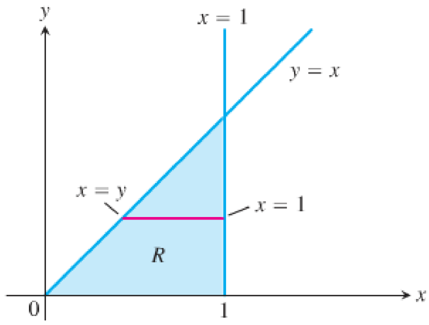
Example:
Find the volume of the prism $\iint_R (3-x-y)\,\mathrm{d}A$ where R is the region bounded by the x-axis and the lines $x=1$ and $y=x$.



(a)



(b)



(c)

The region of integration in the x-y plane and the volume defined by $z=3-x-y$ are shown in the figure. In order to do the double integral we will first consider the approach where we fix the value of x and do the y integral. We have

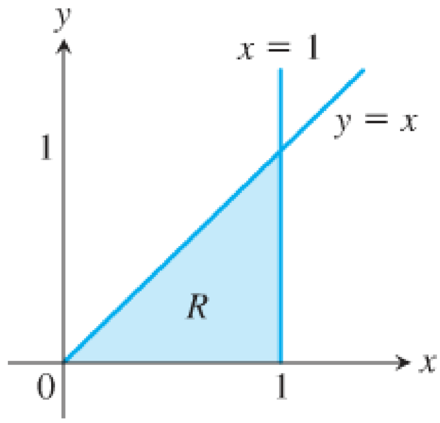
$$\begin{aligned}\int_R\int(3-x-y)\,\mathrm{d}A &= \int_{x=0}^{x=1}\int_{y=0}^{y=x}(3-x-y)\,\mathrm{d}y\,\mathrm{d}x=\int_0^1\left[3y-xy-\frac{y^2}{2}\right]_{y=0}^{y=x}\mathrm{d}x\\ &= \int_0^1\left(3x-\frac{3x^2}{2}\right)\,\mathrm{d}x=\left[\frac{3x^2}{2}-\frac{x^3}{2}\right]_0^1=1\,.\end{aligned}$$

We can also change the order of the integration where we fix the value of y and do the x integral. We have

$$\begin{aligned}\int_R\int(3-x-y)\,\mathrm{d}A &= \int_{y=0}^{y=1}\int_{x=y}^{x=1}(3-x-y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^1\left[3x-\frac{x^2}{2}-xy\right]_{x=y}^{x=1}\mathrm{d}y\\ &= \int_0^1\left(\left(3-\frac{1}{2}-y\right)-\left(3y-\frac{y^2}{2}-y^2\right)\right)\,\mathrm{d}y\\ &= \int_0^1\left(\frac{5}{2}-4y+\frac{3}{2}y^2\right)\,\mathrm{d}y=\left[\frac{5}{2}y-2y^2+\frac{y^3}{2}\right]_{y=0}^{y=1}=1\,.\end{aligned}$$

In some cases the order of integration can be crucial to solving the problem.

Example:
Calculate $\iint_R (\sin x)/x \, \mathrm{d}A$ where R is the triangle in the x-y plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.



Taking vertical strips (i.e. keeping x fixed and allowing y to vary) gives

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x} \, \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left[y \frac{\sin x}{x} \right]_{y=0}^{y=x} \mathrm{d}x = \int_0^1 \sin x \, \mathrm{d}x$$
$$= [-\cos x]_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1 = 0.4596977\,.$$

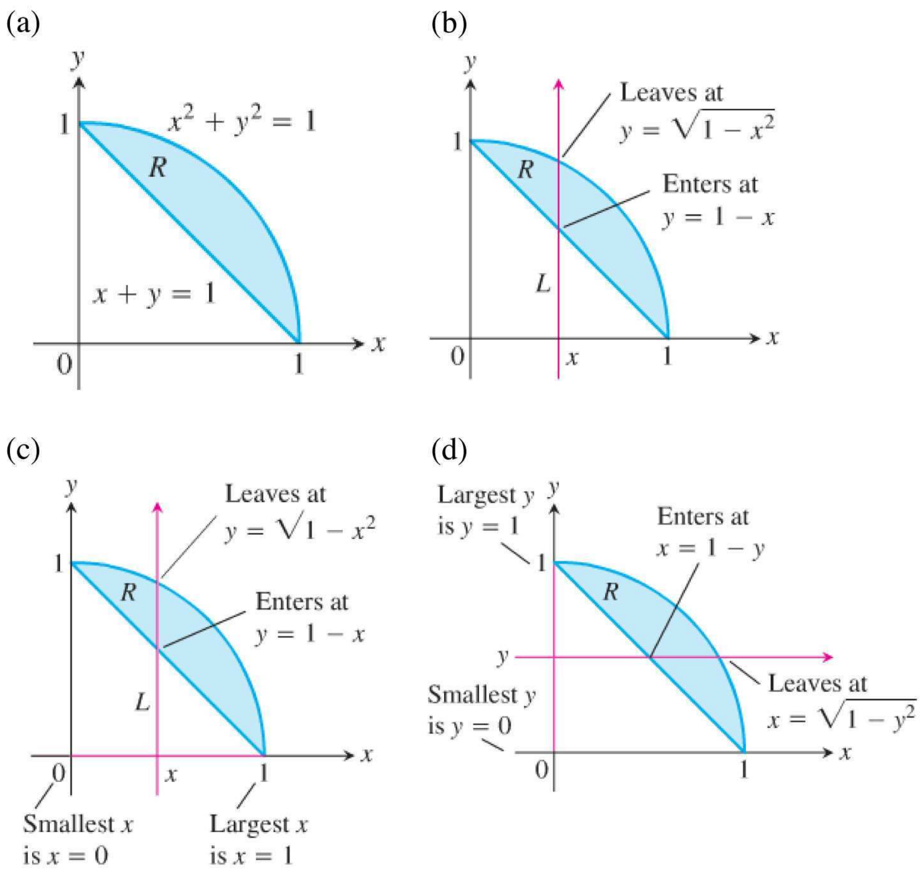
However, if we reverse the order of integration we get

$$\int_0^1 \int_y^1 \frac{\sin x}{x} \, \mathrm{d}x \, \mathrm{d}y$$

and $\int (\sin x)/x \, \mathrm{d}x$ cannot be expressed in terms of elementary functions making the integral difficult to do.
There are always two ways to do a double integral; choose the simpler because the other may be impossible!

A key part of the process of double (and multiple) integration over a region is to find the **limits of the integration**. We can illustrate the procedure by considering the double integral of a function over the region R given by the intersection of the line $x + y = 1$ with the circle $x^2 + y^2 = 1$ (see the picture next page).

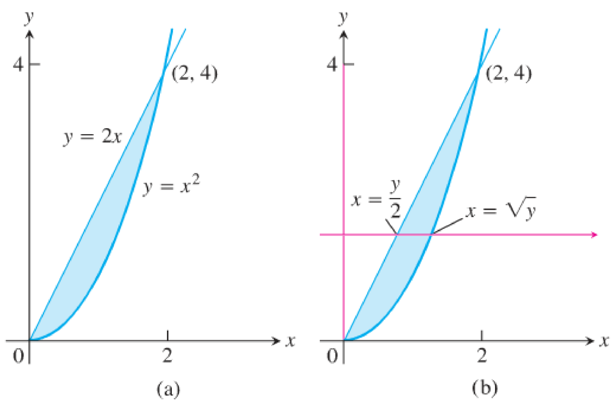
1. **Sketch the region of integration** and label its boundary curves.
2. If we decide to use vertical cross-sections first: **Find the y-limits of integration**. Imagine a vertical line through the region, R, and mark the points where it enters and leaves R. In this case such a line would enter at $y = 1 - x$ and leave at $y = \sqrt{1 - x^2}$.
3. **Find the x-limits of integration**: Choose the x-limits that include all vertical lines through R. In this case the lower limit is $x = 0$ and the upper limit is $x = 1$.
4. This step may not be necessary: **Reversing the order of integration**. Then the x-limits would be from $x = 1 - y$ to $x = \sqrt{1 - y^2}$ and the y-limits from $y = 0$ to $y = 1$.



Example:
Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, \mathrm{d}y\, \mathrm{d}x$$

and write an equivalent integral with the order of integration reversed. Evaluate the integral.



As written, the order of integration would imply that we do the y-integral first, from $y = x^2$ to $y = 2x$, followed by the x-integral from $x = 0$ to $x = 2$. However, we are told to reverse the order of integration. This means we do the x-integration first, from $x = y/2$ to $x = \sqrt{y}$, followed by the y-integral from $y = 0$ to $y = 4$. In other words,

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,\mathrm{d}x\,\mathrm{d}y.$$

We can evaluate the integral using either ordering. Let us revert to the original:

$$\begin{aligned}
\int_0^2 \int_{x^2}^{2x} (4x+2)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^2 \left[4xy+2y\right]_{x^2}^{2x}\,\mathrm{d}x = \int_0^2 \left(8x^2+4x-4x^3-2x^2\right)\,\mathrm{d}x \\
&= \int_0^2 \left(-4x^3+6x^2+4x\right)\,\mathrm{d}x = \left[-x^4+2x^3+2x^2\right]_0^2 \\
&= -16+16+8 = 8\,.
\end{aligned}$$

Note that this example is *not* separable because it is a non-rectangular region (i.e. the limits on the x and y integrals now depend on the region of integration).

Double integrals can also be calculated over unbounded regions.

Example:
Evaluate the integral $\int_0^\infty \int_0^\infty x\,e^{-(x+2y)}\mathrm{d}x\,\mathrm{d}y$.
We have

$$\begin{aligned}
\int_0^\infty \int_0^\infty x\,e^{-(x+2y)}\mathrm{d}x\,\mathrm{d}y &= \int_0^\infty \int_0^\infty e^{-2y} x\,e^{-x}\mathrm{d}x\,\mathrm{d}y \\
&\qquad \text{(integrate by parts with } u = x,\ \mathrm{d}v = e^{-x}\mathrm{d}x) \\
&= \int_0^\infty e^{-2y} \left\{ \left[-x\,e^{-x}\right]_0^\infty - \int_0^\infty \left(-e^{-x}\right)\,\mathrm{d}x \right\}\,\mathrm{d}y \\
&= \int_0^\infty e^{-2y} \left((0-0) + \left[-e^{-x}\right]_0^\infty\right)\,\mathrm{d}y \\
&= \left[-\frac{1}{2}e^{-2y}\right]_0^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}\,.
\end{aligned}$$

Double integrals have the following properties:
Let $f(x,y), g(x,y)$ be continuous on the bounded region R. Then

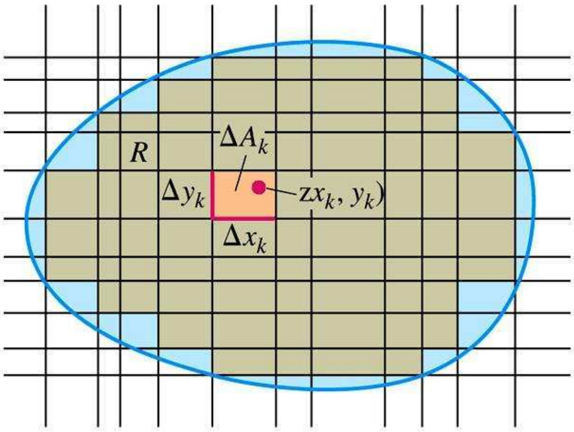
$$\begin{aligned}
\iint_R c\,f(x,y)\,\mathrm{d}A &= c \iint_R f(x,y)\,\mathrm{d}A \quad \text{for any number } c, \\
\iint_R \left(f(x,y) \pm g(x,y)\right)\,\mathrm{d}A &= \iint_R f(x,y)\,\mathrm{d}A \pm \iint_R g(x,y)\,\mathrm{d}A, \\
\iint_R f(x,y)\,\mathrm{d}A &\geq 0 \quad \text{if} \quad f(x,y) \geq 0 \text{ on } R, \\
\iint_R f(x,y)\,\mathrm{d}A &\geq \iint_R g(x,y)\,\mathrm{d}A \text{ if } f(x,y) \geq g(x,y) \text{ on } R, \\
\iint_R f(x,y)\,\mathrm{d}A &= \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_2} f(x,y)\,\mathrm{d}A, \\
&\qquad \text{if } R = R_1 \cup R_2\,,\ R_1 \cap R_2 = \emptyset.
\end{aligned}$$

Area by Double Integration

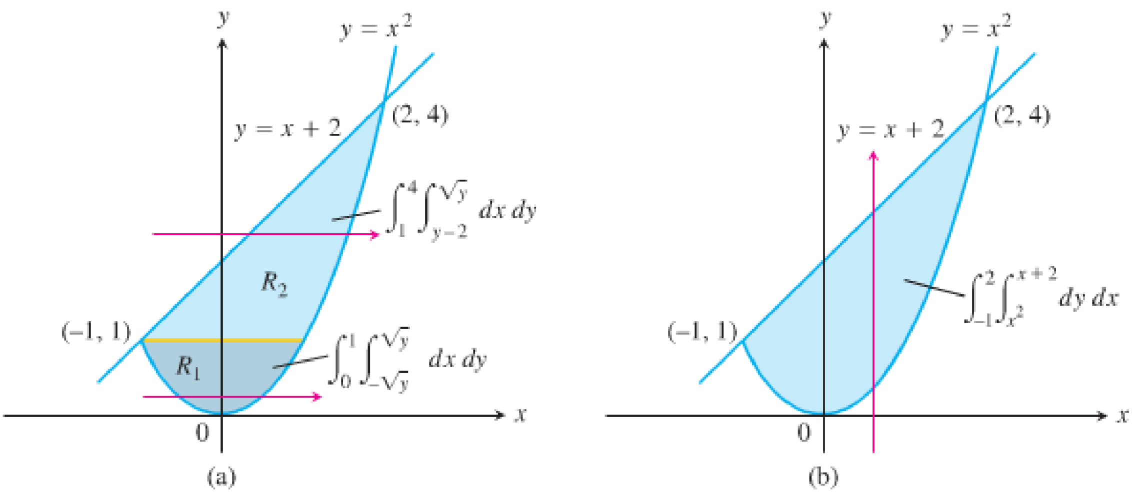
The **area** A of a closed, bounded plane region R is given by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k = \iint_R \mathrm{d}A\,,$$

which is equivalent to calculating $\iint_R f(x,y)\,\mathrm{d}A$ with $f(x,y) = 1$.



Example:
Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$. Determining the points of intersection is essential to determining the limits on the integrations. We can find the points by setting $x^2 = x+2$ which gives $x^2-x-2 = (x+1)(x-2) = 0$, giving $x = -1$ and $x = 2$. The corresponding values of y are $y = 1$ and $y = 4$. So the points of intersection are $(-1, 1)$ and $(2, 4)$.



If we use vertical strips (i.e. fix x and vary y) for the first integral we will not have to split up the region of integration. From the diagram we see that the lower and upper limits for the first integration are therefore $y = x^2$ and $y = x + 2$. This gives

$$\begin{aligned} A &= \int_{-1}^{2}\int_{x^2}^{x+2} \mathrm{d}y\,\mathrm{d}x = \int_{-1}^{2} [y]_{x^2}^{x+2}\,\mathrm{d}x \\ &= \int_{-1}^{2} \left(x + 2 - x^2\right)\,\mathrm{d}x = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}\,. \end{aligned}$$

Double integrals can also be used to find the **average value** of the function $f(x, y)$ over the region R, which is defined to be

$$\langle f\rangle = \text{ave(f)} = \frac{1}{\text{area of } R}\iint_R f(x, y)\,\mathrm{d}A\,.$$

Example:
Find the average value of $f(x, y) = x\,\cos xy$ over the rectangle $R:\ 0 \le x \le \pi,\ 0 \le y \le 1$. The area of the region R is just π, the product of the length of the two sides of the rectangle. We just need to find $\iint_R f(x, y)\,\mathrm{d}A$ and then divide by π.

$$\begin{aligned} \int_0^{\pi}\int_0^{1} x\,\cos xy\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{\pi} [\sin xy]_{y=0}^{y=1}\,\mathrm{d}x \\ &= \int_0^{\pi} (\sin x - 0)\,\mathrm{d}x = [-\cos x]_0^{\pi} = 1 + 1 = 2\,. \end{aligned}$$

Hence $\langle f\rangle = 2/\pi$.