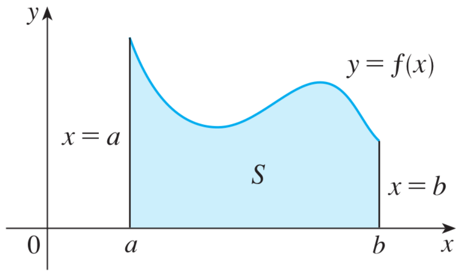
§5 Integrals

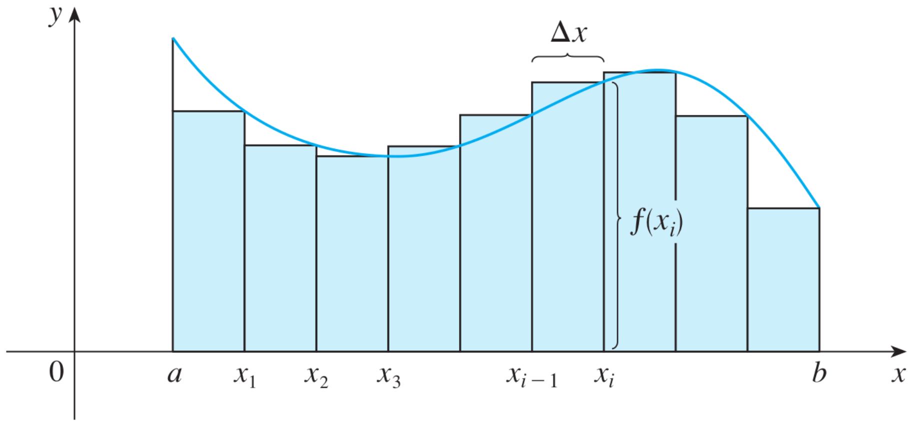
b) Riemann Sum and Definite Integrals

Goal: Given a (continuous) function f on an interval $[a, b]$, compute the area under the graph:



$$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$$

Idea: Split the interval $[a, b]$ in n parts of length $\Delta x = \dfrac{b-a}{n}$ and approximate the area by n rectangles.



Approximation of the area by a sequence of rectangles (the so-called Riemann sum) is:

$$\begin{aligned} S_n =& f(a + \Delta x) \cdot \Delta x + f(a + 2\Delta x) \cdot \Delta x + \ldots \\ & \ldots + f(a + (n-1)\Delta x) \cdot \Delta x + f(a + n\Delta x) \cdot \Delta x \\ =& \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x. \end{aligned}$$

If the limit $\lim_{n\to\infty} S_n$ exists, it is called the *definite integral of the function* f *over* $[a, b]$.

$$\lim_{n\to\infty} S_n = \int_{\underset{\text{lower limit}}{\underset{\uparrow}{a}}}^{\overset{\text{upper limit}}{\overset{\downarrow}{b}}} \underbrace{f(x)}_{\text{integrand}} \underbrace{dx}_{\text{variable of integration}}$$

Theorem. *If f is a continuous function over the interval $[a, b]$, or if f has only a finite number of jump discontinuities, then the definite integral $\int_a^b f(x)\,dx$ exists.*

c) Properties of Definite Integrals

Assume $a < b$ and define

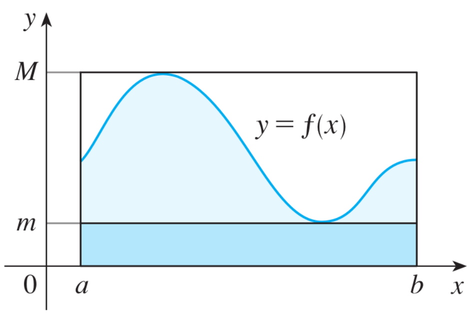
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$
$$\int_a^a f(x)\,dx = 0$$

Rules:

- $\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$
- $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- If $m \le f(x) \le M$ (i.e. m is a lower bound and M is an upper bound of f) for $x \in [a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

(called min-max-inequality)



Example: Find the bounds of $\int_0^1 f(x)\,dx$ for $f(x) = \dfrac{1}{\sqrt{1+x^4}}$ on $[0, 1]$.

Here, $f(x)$ is decreasing (e.g. $f'(x) < 0$ for $x \in (0, 1)$. That means $f(0) \ge f(x) \ge f(1)$ on $x \in [0, 1]$.

Hence ...

and

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1-0) \leq \int_0^1 f(x)\,dx \leq 1 \cdot (1-0) = 1$$

$$\frac{1}{\sqrt{2}} \leq \int_0^1 \frac{1}{1+x^4}\,dx \leq 1$$

which yields

$$.7070 < \int_0^1 \frac{1}{1+x^4}\,dx \leq 1.$$

- If $f(x) \geq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Example: Compute $\int_0^1 x\,dx$ using the definition of the integral in terms of the Riemann sum:

$$f(x) = x, \qquad a = 0, \qquad b = 1, \qquad \Delta x = \frac{b-a}{n} = \frac{1}{n}$$

$$\begin{aligned}
S_n &= f(\underset{\substack{\| \\ 0}}{a} + \Delta x)\Delta x + f(\underset{\substack{\| \\ 0}}{a} + 2\Delta x)\Delta x + \ldots \\
&\ldots + f(\underset{\substack{\| \\ 0}}{a} + (n-1)\Delta x)\Delta x + f(\underset{\substack{\| \\ 0}}{a} + n\Delta x)\Delta x \\
&= 0 + \Delta x \cdot \Delta x + 2\Delta x \cdot \Delta x + 3\Delta x \cdot \Delta x + \ldots \\
&\ldots + (n-1)\Delta x \cdot \Delta x + n\Delta x \cdot \Delta x \\
&= (\Delta x)^2 \big(\underbrace{1 + 2 + 3 + \ldots + (n-1) + n}_{\frac{1}{2}n(n-1)} \big) \\
&= \frac{1}{n^2} \cdot \frac{1}{\cancel{n}} \cdot \frac{1}{2}\cancel{n}(n+1) \\
&= \frac{1}{2}\left(1 + \frac{1}{n}\right).
\end{aligned}$$

$$\begin{aligned}
\int_0^1 x\,dx &= \lim_{n\to\infty} S_n \\
&= \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \\
&= \frac{1}{2}(1+0) = \frac{1}{2}
\end{aligned}$$

Check: This is easily verified to be also the area of the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

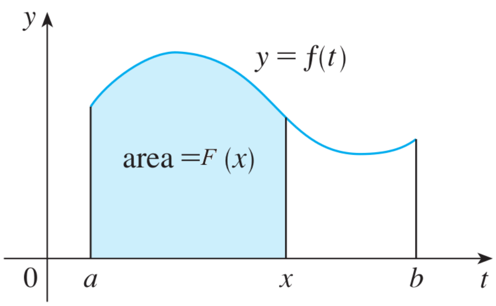
d) Antiderivatives and the Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus). *Consider a continuous function f on $[a,b]$. Then*

$$F(x) = \int_a^x f(t)\,dt$$

defines a continuous function F for $x \in [a,b]$, which is also differentiable for $x \in (a,b)$, and

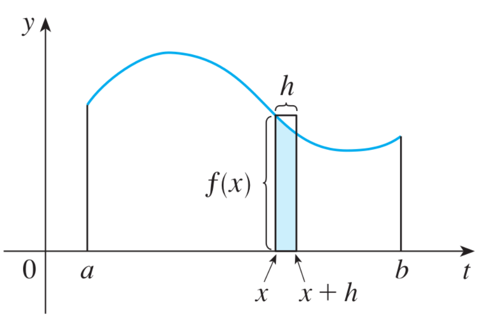
$$F'(x) = \frac{dF(x)}{dx} = \frac{d\int_a^x f(t)\,dt}{dx} = f(x).$$



Reason:
Recall

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \approx f(x)\cdot h \end{aligned}$$



Thus:

$$\frac{F(x+h) - F(x)}{h} \approx f(x), \text{ for small } h$$

and (letting $h \to 0$)

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

Definition. *A differentiable function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$.*

$$\begin{aligned} \textbf{Example: } & f(x) = 3x^2 \\ \text{Antiderivative: } & F(x) = x^3 \\ & (\text{since } F'(x) = \frac{dx^3}{dx} = 3x^2 = f(x).) \end{aligned}$$

$F(x) = x^3 + c$ (where $c \in \mathbb{R}$ is a constant) is an antiderivative as well, since

$$F'(x) = \frac{d(x^3 + c)}{dx} = 3x^2 = f(x).$$

In general: If $F(x)$ is an antiderivative of $f(x)$ (i.e. $F'(x) = f(x)$), then $F(x) + c$ is an antiderivative as well.

Remark. *It is easy to show that $F(x) + c$ is an antiderivative of $f(x)$ (assuming $F'(x) = f(x)$). It is slightly more subtle to prove that there are no further antiderivatives, i.e., all antiderivatives can be written as $F(x) + c$.*

$$\begin{aligned} \textbf{Example: } & f(x) = \cos x \\ \text{Antiderivative: } & F(x) = \sin x) + c \\ & (\text{since } F'(x) = \frac{d(\sin x) + C)}{dx} = \cos x = f(x).) \end{aligned}$$

(since $F'(x) = \dfrac{d(\sin x) + c)}{dx} = \cos x = f(x)$)

Theorem (Second Fundamental Theorem of Calculus). *If f is continuous over $[a, b]$ and F is any antiderivative of f, then*

$$\begin{aligned} \int_a^b f(x)\,dx &= F(b) - F(a) \\ &= F(x)\Big|_{x=a}^{x=b} \end{aligned}$$

Example: $\displaystyle\int_0^\pi (x + \sin x))\,dx$

$$f(x) = x + \sin x), \qquad a = 0, \qquad b = \pi$$

$$\begin{aligned} F(x) &= \frac{1}{2}x^2 + (-\cos x) + \not{c} \\ &= \frac{1}{2}x^2 - \cos x + \not{c} \\ \text{(since } F'(x) &= \frac{1}{2}2x - (-\sin x)) = f(x).) \end{aligned}$$

Thus:

$$\begin{aligned} &\int_0^\pi (x + \sin x))\, dx \\ &= \left(\frac{1}{2}x^2 - \cos x\right)\Big|_{x=0}^{x=\pi} \\ &= \frac{1}{2}\pi^2 - \underbrace{\cos(\pi)}_{-1} - \left(\frac{1}{2}0^2 - \underbrace{\cos(0)}_{1}\right) \\ &= \frac{1}{2}\pi^2 + 2 \end{aligned}$$

Notation: Let F denote an antiderivative of the function f. We denote the general antiderivative by

$$F(x) + C = \int f(x)\, dx,$$

and we call $\int f(x)\, dx$ the *indefinite integral* of f and $\int_a^b f(x)\, dx$ the *definite integral* of f (over $[a, b]$).

Example:

$$\int (x + \sin x))\, dx = \frac{1}{2}x^2 - \cos x + c$$

is the indefinite integral of $x + \sin x)$ and

$$\int_0^\pi (x + \sin x))\, dx = \frac{1}{2}\pi^2 + 2$$

is the indefinite integral of $x + \sin x)$ over $[0, \pi]$.

Example:
Compute the definite integral

$$\int_0^\pi f(x)\, dx,$$

where

$$f(x) = \begin{cases} \cos x & \text{if } 0 \le x \le \pi/2 \\ \sin x) & \text{if } \pi/2 < x \le \pi. \end{cases}$$

Solution:

$$\begin{aligned}
\int_0^{\pi} f(x)\,dx &= \int_0^{\pi/2} \underbrace{f(x)}_{\cos x}\,dx + \int_{\pi/2}^{\pi} \underbrace{f(x)}_{\sin x)}\,dx \\
&= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} \sin x)\,dx \\
&= \sin x)\Big|_{x=0}^{x=\pi/2} + (-\cos x)\Big|_{x=\pi/2}^{x=\pi} \\
&= \underbrace{\sin(\pi/2)}_{1} - \underbrace{\sin(0)}_{0} - \underbrace{\cos(\pi)}_{-1} + \underbrace{\cos(\pi/2)}_{0} \\
&= 2.
\end{aligned}$$

Example: $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$

Here, $f(x) = \frac{1}{\sqrt{1-x^2}}$

$$F(x) = ?, \qquad F'(x) = \frac{1}{\sqrt{1-x^2}}$$

(Recall that $\displaystyle\frac{d\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}}$)

So $F(x) = \arcsin x) = \sin^{-1}(x)$.

$$\begin{aligned}
\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx &= \arcsin x)\Big|_{x=0}^{x=1/2} \\
&= \underbrace{\arcsin(1/2)}_{\pi/6} - \underbrace{\arcsin(0)}_{0} \\
&= \pi/6 \qquad\qquad (\text{since } \sin(\pi/6) = 1/2).
\end{aligned}$$

Integral Methods (or Methods of Integration)

(i) Substitution Method

Let $F(x)$ denote an antiderivative of $f(x)$, i.e., $F'(x) = f(x)$. Then,

$$\int f(x)dx = F(x) + c$$

Consider the composition $F(g(x))$, then the chain rule tells us that

$$\begin{aligned}\frac{dF(g(x))}{dx} &= F'(g(x)) \cdot g'(x) \\ &= f(g(x))g'(x).\end{aligned}$$

That means the antiderivative of $f(g(x)) \cdot g'(x)$ is given by $F(g(x))$. Hence,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

Using the abbreviation $u = g(x)$, $\frac{du}{dx} = g'(x)$ we have that

$$\int f(g(x))g'(x)dx$$

$$= \int f(u)\frac{du}{dx}dx \qquad = \int f(u)du = F(u) + c.$$

Theorem. *If $u = g(x)$ is a differentiable function whose range is an interval I, and if f is a continuous function on I then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Example: Evaluate

$$\int \frac{2x}{\sqrt{x^2+5}}dx$$

Substitution: $u = x^2 + 5$. Then $\frac{du}{dx} = 2xdx$. So,

$$\begin{aligned}\int \frac{2x}{\sqrt{x^2+5}}dx &= \int \underbrace{\frac{1}{\sqrt{x^2+5}}}_{1/\sqrt{u}} \underbrace{2xdx}_{du} \\ &= \int \frac{1}{\sqrt{u}}du \\ &= 2\sqrt{u} + c \qquad \text{(back substitution of } u = x^2 + 5 \text{ gives)} \\ &= 2\sqrt{x^2+5} + c.\end{aligned}$$

Remark: Substitution rule for definite integrals, $u = g(x)$ is

$$\int_{x=a}^{x=b} f(g(x)) \cdot g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du.$$

Example: Evaluate

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Substitution: Let $u = x^3 + 1$. Then $du = 3x^2\,dx$.

When $x = 1$, $u = g(1) = 2$ and when $x = -1$, $u = g(-1) = 0$.

$$\begin{aligned}
&\int_{-1}^{1} \sqrt{x^3+1}3x^2dx = \\
&\qquad = \int_0^2 \sqrt{u}du \quad \text{(since } x = -1 \text{ implies } u = 0 \text{ and } x = 1 \text{ implies } u = 2\text{)} \\
&\qquad = \frac{2}{3}u^{3/2}\Big|_{u=0}^{u=2} \\
&\qquad = \frac{2}{3}2^{3/2} - 0 = \frac{4}{3}\sqrt{2}.
\end{aligned}$$

Example: Find

$$\int \frac{\cos x}{\sqrt{2+\sin x)}}dx$$

Substitution:

$$\begin{aligned}
u &= 2 + \sin x) \\
du &= \cos x dx
\end{aligned}$$

$$\int \underbrace{\frac{1}{\sqrt{2+\sin x)}}}_{\frac{1}{\sqrt{u}}} \cdot \underbrace{\cos x dx}_{du} = \int \frac{1}{\sqrt{u}}du$$

$$= 2\sqrt{u} + c = 2\sqrt{2+\sin x)} + c.$$

Example: Find

$$\int \frac{1}{\sqrt{x-x^2}}dx$$

Substitution:

$$\begin{aligned} u &= ? \\ u = \sqrt{x} &\Rightarrow u^2 = x \\ du &= \frac{1}{2\sqrt{x}}dx \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sqrt{x-x^2}}dx &= \int \frac{1}{\sqrt{x(1-x)}}dx \\ &= \int \frac{1}{\sqrt{x}}\frac{1}{\sqrt{1-x}}dx \\ &= 2\int \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}}dx \\ &= 2\int \underbrace{\frac{1}{\sqrt{1-x}}}_{\frac{1}{\sqrt{1-u^2}}}\underbrace{\frac{1}{2\sqrt{x}}dx}_{du} \\ &= 2\int \frac{1}{\sqrt{1-u^2}}du \\ &= 2 \cdot \arcsin(u) + c \\ &= 2 \cdot \arcsin(\sqrt{x}) + c. \end{aligned}$$

(ii) Integration by Parts

Recall the product rule

$$\frac{d(f(x)g(x))}{dx} = f(x)g'(x) + f'(x)g(x)$$

That means $f(x)g(x)$ is an antiderivative of $f(x)g'(x) + f'(x)g(x)$, i.e.,

$$\begin{aligned} f(x)g(x) + c &= \int (f(x)g'(x) + f'(x)g(x))dx \\ &= \int f(x)g'(x)dx + \int f'(x)g(x)dx \end{aligned}$$

Hence:

$$\int f(x)g'(x)dx = f(x)g(x) + c - \int f'(x)g(x)dx$$

Notice that the constant c cancels in a definite integral! So, we write

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx.$$

Alternatively, letting $u = f(x)$, $v = g(x)$, we have $du = f'(x)dx$, $dv = g'(x)dx$ which implies

$$\int u dv = uv - \int v du.$$

Example: Find

$$\int x^2 e^x \, dx.$$

Letting $u = x^2$, $dv = e^x \, dx$, we get $du = 2x dx$ and $v = e^x$.

So, integration by parts yields,

$$\begin{aligned}\int x^2 e^x \, dx &= uv - \int v du \\ &= x^2 e^x - 2 \int x e^x dx\end{aligned}$$

This integral requires another round of integration by parts, since we have $\int x e^x dx$ in the second integral.

Again, letting $u = x$, $dv = e^x \, dx$, we get $du = dx$ and $v = e^x$.

(Notice that we are using the same u and v in both rounds of the integration by parts, to avoid the clutter in the notation.)

So, we get

$$\begin{aligned}&= x^2 e^x - 2 \left(x e^x - \int e^x dx \right) \\ &= x^2 e^x - 2x e^x + 2 \int e^x dx \\ &= x^2 e^x - 2x e^x + 2e^x + c.\end{aligned}$$

Example: For $x > 0$, find

$$\int \ln x dx.$$

Letting $u = \ln x$, $dv = dx$, we get $du = \frac{1}{x} dx$ and $v = x$.

So, integration by parts yields,

$$\begin{aligned}\int \ln |x| dx &= (\ln x)x - \int x \cdot \frac{1}{x} dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + c.\end{aligned}$$

Example: Find
$$\int e^x \cos x dx.$$

Letting $u = \cos x$, $dv = e^x dx$, we get $du = -\sin x)dx$ and $v = e^x$.

So, integration by parts yields,
$$\begin{aligned}\int e^x \cos x dx &= e^x \cos x - \int e^x(-\sin x))dx \\ &= e^x \cos x + \int e^x \sin x)dx\end{aligned}$$

This integral requires another round of integration by parts, since we have $\int e^x \sin x)dx$ in the second integral.

Again, letting $u = \sin x)$, $dv = e^x\, dx$, we get $du = \cos x dx$ and $v = e^x$.

(Notice that we are using the same u and v in both rounds of the integration by parts, to avoid the clutter in the notation.)

So, we get
$$\begin{aligned}\int e^x \cos x dx &= e^x \cos x + \left(e^x \sin x) - \int e^x \cos x dx\right) \\ 2\int e^x \cos x dx &= e^x \cos x + e^x \sin x)\end{aligned}$$

which implies
$$\int e^x \cos x dx = \frac{e^x(\sin x) + \cos x)}{2}.$$