Lecture 1 - Probability Basics

A student in ACLC hall is randomly selected and the student is from COE: College of Engineering, COSAM: College of Sciences and Math or oth: Other

Let X = college of the student

X = COE   outcome ω

X = COSAM

X = oth

Sample Space

$\Omega = \{COE, COSAM, oth\}$

- An event, A, is any subset of Ω: $A \subseteq \Omega$
- Def'n: Probability P: subsets of $\Omega \longrightarrow \mathbb{R}$, actually $[0,1]$
- Suppose $P(X=COE) = 0.5$, $P(X=COSAM) = 0.3$, $P(X=oth) = 0.2$

$$\sum_{i \in \Omega} P(X=i) = 1 \checkmark$$

If X = COE, then $P(X = COSAM) = 0$

Mutually exclusive (ME) Events:

$E_1, E_2, \ldots, E_n$ are mutually exclusive if

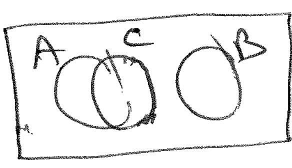
$E_i \cap E_j = \emptyset$ for all $i \neq j$

"Is $A \cap C = \emptyset$?"

Note: ① $A \cap B = \emptyset$ and $B \cap C = \emptyset$, $A \cap C \overset{?}{=} \emptyset$

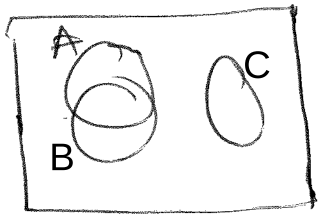
Not necessarily

see

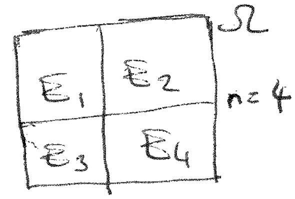


(2) If $A \cap B \cap C = \emptyset$, then $A \cap B \overset{?}{=} \emptyset$

Not necessarily!



- <u>Def'n</u> $E_1, \ldots, E_n$ is a <u>partition</u> of Ω or
  " <u>partition</u> Ω,



if (i) events are mutually exclusive
and (ii) $\bigcup_{i=1}^{n} E_i = \Omega$ $\quad E_1 \cup E_2 \cup \cdots \cup E_n = \Omega$

<u>Ex 1:</u> {CoE}, {CoSAM}, {oth} is a partition (of Ω)
{CoE, oth}, {CoSAM} " another " " " .

<u>Assigning Probabilities</u>

<u>Method 1:</u> Set Theory

$P(X \in \{\text{CoE or CoSAM}\}) = 0.8$

$P(\text{CoE} \cup \text{CoSAM}) = P(\text{CoE}) + P(\text{CoSAM}) - \cancel{P(\text{CoE} \cap \text{CoSAM})} \to 0$

(inclusion-exclusion)

$= 0.5 + 0.3 - 0 = 0.8.$

In general $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

<u>ME Rule:</u> (1) If E_1 & E_2 are ME, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

(2) If $E_1, E_2, \ldots, E_n$ are ME, then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

<u>Def'n</u> P: Subsets of $\Omega \longrightarrow \mathbb{R}$

(i) $P(A) \geq 0$ for all $A \subseteq \Omega$

(ii) $P(\Omega) = 1$

(iii) $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ for ME events $E_i \subseteq \Omega$

<u>Applications</u>: (1) $P(A^c) = 1 - P(A)$

<u>Sol'n:</u> $P(A \cup A^c) = P(\Omega) = 1$

Since A & A^c are ME, $P(A \cup A^c) = P(A) + P(A^c)$

So, $P(A) + P(A^c) = 1 \Rightarrow P(A^c) = 1 - P(A)$.

(2) Union Bound: $P(A \cup B) \leq P(A) + P(B)$ for any events A & B

<u>Sol'n:</u> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

since $P(A \cap B) \geq 0$, we have

$P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

$\Rightarrow P(A \cup B) \leq P(A) + P(B)$

In general, $P(A \cup B \cup \cdots) \leq P(A) + P(A) + \cdots$

<u>Def'n:</u> $\underbrace{A \text{ and } B \text{ are independent}}_{A \perp B}$ iff $P(A \cap B) = P(A)P(B)$

(1) A, B, C are pairwise indep. if $A \perp B$, $A \perp C$, $B \perp C$

(2) A, B, C " mutually " if " " " & $P(A \cap B \cap C) = P(A)P(B)P(C)$

<u>Method 2</u> Counting (works for equally likely outcomes)

Assume a student's getting A, B or C are equally likely.

$P(\text{grade better than } C) = 2/3$

In general, $P(E) = \dfrac{\#\text{ of outcomes favorable to } E}{|\Omega|}$

$= \dfrac{|E|}{|\Omega|}$

X & Y are two students & get grades A, B, C in pairs equally likely

$P(X=A \cap Y=A) \stackrel{?}{=} P(X=A)\,P(Y=A)$ ✓ $X=A$ & $Y=A$ are indep.

$\{A,A\}, \{A,B\} \ldots \quad 1/9 \quad \frac{1}{3} \times \frac{1}{3}$

Ex. $\{A,B,C\}$ equally likely, 10 students

P (exactly 3 students get A)

$= \dfrac{\binom{10}{3} 2^7}{3^{10}} \qquad |\Omega| = 3^{10}$

$\underbrace{A\ A\ A}_{3 \text{ get } A}\ \underbrace{B/C\ B/C \ldots B/C}_{2^7}$

$\{A, A, \ldots\}$

$\{A, A, A, B, \ldots B\}$ $\quad = \binom{10}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^7$ ✓

$\{A, A, B, B \ldots C, B, A\}$

Extension 1: P(at most 3 stds get A)

$= \binom{10}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{10} + \binom{10}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^9 + \binom{10}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^8$

$+ \binom{10}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7$

Extension 2: P (at least 4 stds get A)

$= 1 - P(\text{at most 3 stds get A})$

Qu: Suppose $A \perp B$. Can A & B be ME?

Ans: $P(A \cap B) = P(A)P(B)$ since $A \perp B$

For A & B to be ME, $A \cap B = \emptyset$ must hold, which would imply $P(A \cap B) = 0$

So, $P(A)P(B) = 0$ iff $P(A) = 0$ or $P(B) = 0$.

Thus A & B can be ME, if $A = \emptyset$ or $B = \emptyset$

Def'n RND(a, b) generates all integers between a & b (inclusive) with equal prob.

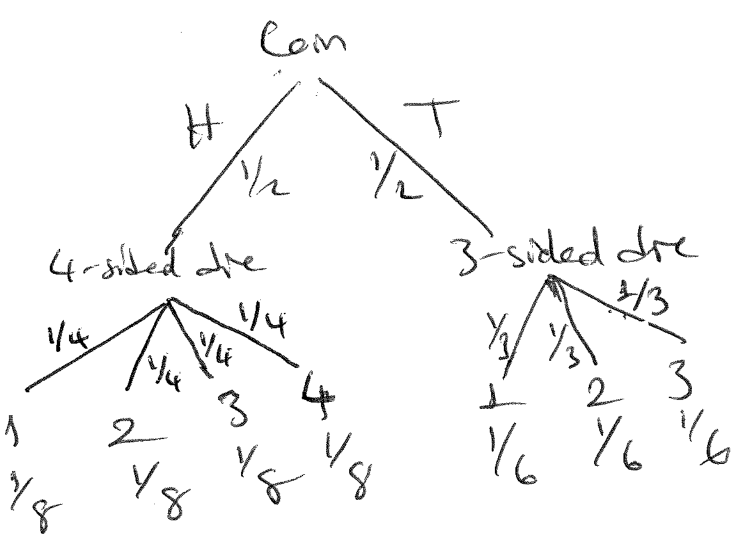
$$RND(1,3) = \begin{cases} 1 & \text{wp } 1/3 \\ 2 & \text{" "} \\ 3 & \text{" "} \end{cases}$$

die: RND(1, 6)
coin: RND(0, 1)

Method 3 Trees

Ex: 1. toss a fair coin

2. If coin shows Heads, roll a 4-sided fair die
else " " 3-" " "



① P(getting a 4 on 4-sided die) = ?

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

② P(die ≥ 3)

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{6} = \frac{5}{12}$$

Notes: (1) Branches form a partition

(2) Could use RND(0,1) → RND(1,3), RND(1,4)

Ex 1 We have 100 indep. tosses of an unbiased coin. If 99 tosses result in a H, what is the prob

$P(100^{th}\ \text{toss} = H) = ?\quad \frac{1}{2}$

Ex 2: Roll two unbiased dice, D_1 & D_2, independently.

(1) Is $D_1 + D_2 = 5 \perp D_2 = 6$? No,

(2) " " " " $D_2 = 5$? No

(3) " " " " $D_2 = 4$? No

(4) " " " " $D_2 = y$: $y \in \{1, 2, \dots, 6\}$ Yes

(5) " " $= 3 \perp D_2 = y$ " No

(6) " " $= 7 \perp D_2 = 3$ Yes!

$$\underbrace{P(D_1 + D_2 = 7 \cap D_2 = 3)}_{1/36} \overset{?}{=} \underbrace{P(D_1 + D_2 = 7)}_{\frac{6}{36} = 1/6} \times \underbrace{P(D_2 = 3)}_{1/6}$$

Qu: $D_1 + D_2 = 20 \perp D_2 = 4$ Yes!

- For any event A, $A \perp \emptyset$

Sol'n: $P(A \cap \emptyset) \overset{?}{=} P(A)\, P(\emptyset)$

$P(\emptyset) \overset{?}{=} P(A)\ 0$

$0 = 0$ ✓.

- $A \perp B \Rightarrow A \perp B^c$

" $\Rightarrow A^c \perp B$

" $\Rightarrow A^c \perp B^c$

Recall {CoE, COSAM, oth}

Qn: 6 students are randomly selected; 2 from each of CoE, COSAM, oth colleges.

(1) P(first std from CoE) = $P(S_1 = \text{CoE}) = ?$

{CoE, CoE, COSAM, COSAM, oth, oth} $\frac{2}{6} = \frac{1}{3}$

or $|\Omega| = \frac{6!}{2!\,2!\,2!} = 90$ & $|S_1 = \text{CoE}| = \frac{5!}{2!\,2!} = 30$

$\Rightarrow P(S_1 = \text{CoE}) = 30/90 = \frac{1}{3}$

(2) P(first 3 stds from different colleges)
$= P(S_{1-3} \in \{\text{CoE, COSAM, oth}\}) = ?$

$\underbrace{\text{CoE, COSAM, oth}}_{3!}$, $\underbrace{\text{CoE, COSAM, oth}}_{3!}$ $= 36$ $\Rightarrow \frac{36}{90} = \frac{2}{5}$

$S_1 \quad S_2 \quad S_3$ remaining 3 stds

or $P(S_{1-3} \in \{\text{CoE, COSAM, oth}\})$

$= \frac{2}{6} \times \frac{2}{5} \times \frac{2}{4} \times 3! = \frac{2}{5}$

(3) $P(S_{1-3} \in \{\text{CoE, COSAM, oth}\} \cap S_6 = \text{oth})$

$= 3!\,.\,2!\,/90 = 2/15$

$\underbrace{\text{CoE, COSAM, oth}}_{3!}$ | $\underbrace{\text{CoE, COSAM}}_{2!}$, oth

$S_1 \quad S_2 \quad S_3$ | $S_4 \quad S_5$ $\hat{S}_6$

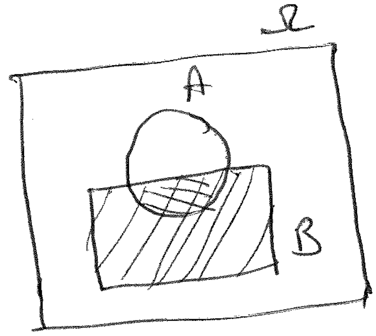
or tree approach

$P(S_{1-3} \in \{coE, cosAM, oth\} \cap S_6 = oth)$

$= P(S_{1-3} \in \{coE, cosAM, oth\}) \times P(S_6 = oth \text{ given } S_{1-3} \in \{coE, cosAM, oth\})$

$= \frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$

Conditional Probability



$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

"prob of A given B"

$\Rightarrow P(A \cap B) = P(A|B)P(B)$

Similarly $P(B|A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(A \cap B) = P(B|A)P(A)$

- $P(A|B) \neq P(B|A)$ unless $P(A) = P(B)$
- If $A \perp B$, then $P(A \cap B) = P(A)P(B) = P(B)P(A|B)$
  $\Rightarrow P(A|B) = P(A)$.
- $P(\cdot|B)$ is a valid prob. function (verify)

Chain Rule: $P(A \cap B) = P(A|B)P(B)$

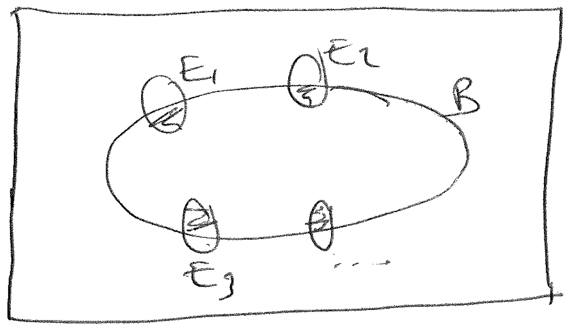
$P(A \cap B \cap C) = P(A|(B \cap C))\, P(B \cap C)$

$= P(A|(B \cap C))\, P(B|C)\, P(C)$

If $E_1, E_2, \ldots$ are ME, rules of prob $\Rightarrow$

$P(\cup E_i) = \sum P(A_i)$

$\Rightarrow P(\cup(E_i|B)) = \sum P(A_i|B)$



Total Probability (TP) Rule:

If $\{E_1, E_2, \dots, E_n\}$ forms a partition of Ω and $A \subseteq \Omega$ is some event, then

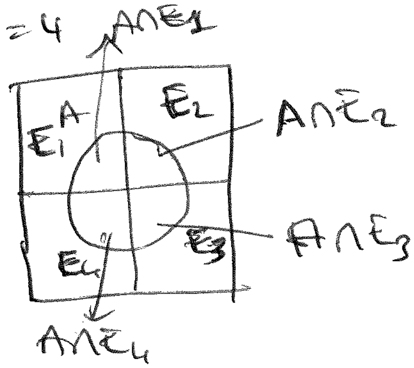
$$P(A) = \sum_{i=1}^{n} P(A|E_i)\,P(E_i)$$

Sol'n $P(A) = P(A \cap \Omega) = P(A \cap (\bigcup_{i=1}^{n} E_i))$

$$= P(\bigcup_{i=1}^{n} A \cap E_i).$$

$$\text{ME} \; \Big(= \sum_{i=1}^{n} P(A \cap E_i) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} \sum_{i=1}^{n} P(A|E_i)\,P(E_i)$$

with $n = 4$



Bayes Thm:
(Simple Version)

Recall $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$\Rightarrow P(A \cap B) = P(A|B)\,P(B)$$

& $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$

$$\Rightarrow P(A \cap B) = P(B|A)\,P(A)$$

$$\Rightarrow P(A|B)\,P(B) = P(B|A)\,P(A) \Rightarrow P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

(general version) If $\{E_1, E_2, \ldots, E_n\}$ forms a partition of Ω and A is any event, then

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{\sum_{j=1}^{n} P(A|E_j)P(E_j)}$$

Pf. $P(E_i|A) = \dfrac{P(A\cap E_i)}{P(A)} \overset{\text{chain rule}}{\underset{\text{TP rule}}{=}} \dfrac{P(A|E_i)\,P(E_i)}{\sum_{j=1}^{n} P(A|E_j)P(E_j)}$

Ex: H: healthy, testing people for the disease
D: diseased (i.e. not healthy)

$P(+|D) = 0.99 \Rightarrow P(-|D) = 0.01$

$P(-|D) = 0.99 \Rightarrow P(+|H) = 0.01$

There is a $1/1000$ chance of catching the disease, i.e. $P(D) = 1/1000$

Qu: $P(D|+) = ?$

$$P(D|+) = \frac{P(+|D)\,P(D)}{P(+|D)\,P(D) + P(+|H)\,P(H)}$$

$$= \frac{0.99 \times 1/1000}{0.99 \times 1/1000 + 0.01 \times 999/1000} = \frac{0.99}{0.99 + 9.99} \approx \frac{1}{11}$$

Paradox: A couple has 2 children and at least one them is a girl.

Q: Prob(both children are girls | at least 1 is a girl) = ?

Sol'n P(both are girls | ≥ 1 is a girl)

BB, BG, GB, GG

$$= \frac{P(\text{both } G \cap \geq 1\, G)}{P(\geq 1\, G)}$$

$$= \frac{P(GG)}{P(\geq 1\, G)} = \frac{1/4}{3/4} = \frac{1}{3} \quad (\text{not } 1/2)$$

Conditional Independence:

Event A & B are $\perp$ conditional on the event C
$\Leftrightarrow P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$

Notes:

(i) $\perp \not\Rightarrow$ Cond $\perp$.

Ex: $A: D_1 = 4$ $B: D_2 = 4$, $A \perp B$

But with $C: D_1 + D_2 = 6$, $A|C \not\perp B|C$

verify! $p(A \cap B \mid C) = \dfrac{P(A \cap B \cap C)}{P(C)} = \dfrac{P(\emptyset)}{P(C)} = 0$

but $P(A \mid C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{P(D_1 = 4, D_2 = 2)}{P(D_1 + D_2 = 6)} = \dfrac{1/36}{5/36} = \dfrac{1}{5}$

(1,5) (5,1), (2,4) (4,2), (3,3)

likewise $p(B \mid C) = 1/5$.

& $0 \neq \frac{1}{5} \times \frac{1}{5}$

(ii) Conditional $\perp \not\Rightarrow \perp$

Ex: $A|C: S_1$ carries a jacket if it snows w.p. 0.999

$B|C: S_2$ " " " " " " 0.980

C: It is snowing

Assume $A|C \perp B|C$ i.e. $P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$
$= 0.999 \times 0.980$

But this has no implication on $P(A \cap B) = P(A)P(B)$!