Recap on Prob Basics

1. Conditional Prob: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

2. Chain Rule: $P(A \cap B) = P(A|B)\,P(B)$

3. TP Rule: $P(A) = \sum_{i=1}^{n} P(A|E_i)\,P(E_i)$
   where $\{E_1, E_2, \ldots, E_n\}$ forms a partition

4. Bayes' Thm:
$$P(E_i|A) = \frac{P(A|E_i)\,P(E_i)}{\sum_{j=1}^{n} P(A|E_j)\,P(E_j)}$$
where E_j are as in 3

Lecture 2 - Random Variables - 1: Overview & Discrete RVs

Def'n: A random variable (rv) is a real-valued function on the outcomes of an experiment (or on the elements of the sample space
or " " sample points)

$X : \Omega \longrightarrow \mathbb{R}$ i.e. $X(\omega)$, for all $\omega \in \Omega$

outcome (ω), sample space (Ω)

R.V.s
- Discrete rv's
- Continuous rv's

Ex: $X = \begin{cases} 1 & \text{if it rains today} \\ 0 & \text{o.w.} \end{cases}$ $\quad \Omega = \{\omega_1, \omega_2\}$
ω_1: rain $\quad \omega_2$: no rain

<u>Ex:</u> X: # on a fair die roll

$$X = \begin{cases} 1 & \text{wp } 1/6 \\ 2 & \text{"} \\ \vdots & \vdots \\ 6 & \text{"} \end{cases}$$

RVs — Discrete RVs → take on separate countable values

RVs — Continuous RVs → any value in an interval in $\mathbb{R}$

<u>Discrete RVs</u> $X: \Omega \to \mathcal{X} \subseteq \mathbb{R}$ pmf: prob. mass function

pmf: $f_X(x) = P(\{w \in \Omega : X(w) = x\})$

(i) $f_X(x) \geq 0$ for all x

(ii) $\sum_{x \in \mathcal{X}} f_X(x) = 1$

<u>Ex:</u> Two fair coins are tossed: X: # of heads

$$\Omega = \{\underset{w_1}{HH}, \underset{w_2}{HT}, \underset{w_3}{TH}, \underset{w_4}{TT}\}$$

each wp $1/4$
$$\begin{cases} X(w_1) = X(HH) = 2 \\ X(w_2) = X(HT) = 1 \\ X(w_3) = X(TH) = 1 \\ X(w_4) = X(TT) = 0 \end{cases} \Rightarrow \mathcal{X} = \{0, 1, 2\}$$

Then, $P(X=0) = P(X=2) = 1/4$ & $P(X=1) = 1/2$

i.e. $f_X(x) = \begin{cases} 1/4 & x \in \{0, 2\} \\ 1/2 & x = 1 \\ 0 & \text{o.w.} \end{cases}$ ← pmf

Properties of Discrete RVs

$\begin{pmatrix} X: \text{discrete rv} \\ f_X(x): \text{pmf} \end{pmatrix}$

(i) Validity test: $f_X(x) \geq 0$ for all x

2 coin flip example

$\sum_{x \in \mathcal{X}} f_X(x) = 1$ e.g. $\sum_{x \in \{0,1,2\}} f_X(x) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$ ✓

(ii) $E[X] = \sum_{x \in \mathcal{X}} x f_X(x)$ e.g. $E[X] = 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right)$
$= 1$.

mean, expectation, weighted average

(iii) $E[X^i] = \sum_x x^i f_X(x)$: i^{th} moment of X

special case: $i = 2$

$E[X^2] = \sum_x x^2 f_X(x)$ e.g. $E[X^2] = 0^2\left(\frac{1}{4}\right) + 1^2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{4}\right)$
$= \frac{3}{2}$

(iv) $Var(X) = E\left[\underbrace{(X - E[X])^2}_{\geq 0}\right] = E[X^2] - (E[X])^2$ $\Rightarrow Var(X) \geq 0$

e.g. $Var(X) = \frac{3}{2} - 1^2 = \frac{1}{2}$

$SD(X) = \sqrt{Var(X)}$ e.g. $SD(X) = \frac{1}{\sqrt{2}}$.

(v) Cumulative distribution function (cdf)

$F_X(s) \equiv P(X \leq s) = \sum_{x \leq s} f_X(x)$

Notation:
$X \sim F$

e.g. for $s < 0$, $F_X(s) = P(X \leq s) = 0$

" $0 \leq s < 1$, $F_X(s) = P(X \leq s) = P(X=0) = 1/4$

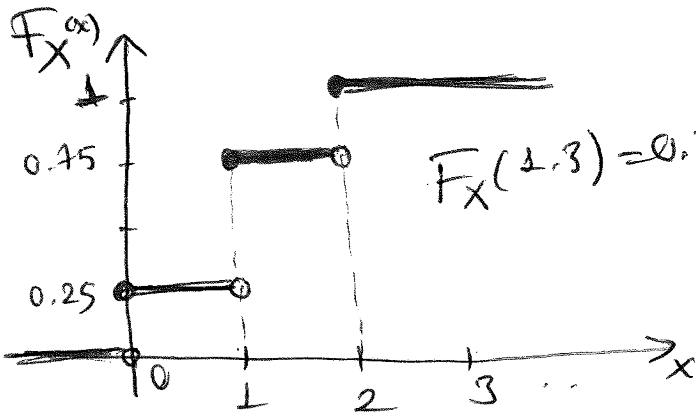
" $1 \leq s < 2$, " " $= P(X=0) + P(X=1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

" $2 \leq s$, " " $= P(X=0) + P(X=1) + P(X=2)$
$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$

i.e.

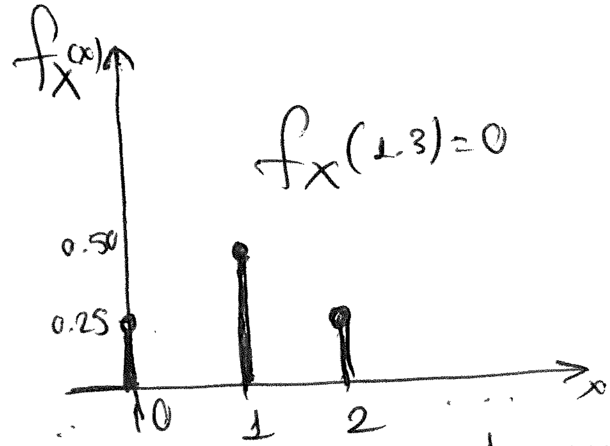
$$F_X(x) = \begin{cases} 0 & x<0 \\ 1/4 & 0\le x<1 \\ 3/4 & 1\le x<2 \\ 1 & x\ge 2 \end{cases}$$

cdf



step fcn, non-decreasing
right continuous, >0 for all $x\ge 0$

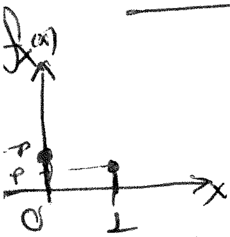
pmf
$$f_X(x) = \begin{cases} 1/4 & x=0,2 \\ 1/2 & x=1 \\ 0 & \text{o.w.} \end{cases}$$



point fcn, discontinuous.
>0 for $x=0,1,2$ only

Note: $P(X>s) = 1-P(X\le s) = 1-F_X(s)$.

- Bernoulli Distr. $X \sim \text{Bernoulli}(p)$ 2 possible outcomes success & failure wp p wp $1-p$



$$X = \begin{cases} 1 & \text{wp } p \\ 0 & \text{" } 1-p \end{cases}$$

eg a biased coin flip with $P(H)=p$ $H \equiv$ success $\equiv 1$, $T \equiv$ failure $\equiv 0$

- $p=0.5$ if coin is fair.

pmf $$f_X(x) = \begin{cases} p & x=1 \\ 1-p & x=0 \\ 0 & \text{o.w.} \end{cases}$$

(i) Validity test $\sum_x f_X(x) = f_X(1) + f_X(0) = p+(1-p) = 1$ ✓

(ii) $E[X] = \sum_x x f_X(x) = 0(1-p) + 1(p) = p$

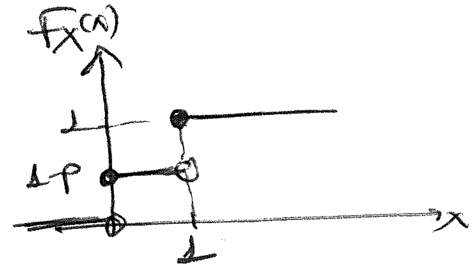
(iii) $E[X^2] = \sum_x x^2 f_X(x) = 0^2(1-p) + 1^2 p = p$

$\Rightarrow$ (iv) $Var(X) = p - p^2 = p(1-p)$

Can write pmf as a formula or function

$$f_X(x) = \begin{cases} p^x(1-p)^{1-x} & x = 0, 1 \\ 0 & \text{o.w.} \end{cases}$$

w/ cdf: $F_X(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \geq 1 \end{cases}$



Ex: $X = I(E)$: indicator of event E

$$I(E) = \begin{cases} 1 & \text{if event E occurs} \\ 0 & \text{" " " does not occur} \end{cases}$$

so, $X = \begin{cases} 1 & \text{wp } P(E) \\ 0 & \text{wp } P(E^c) = 1 - P(E) \end{cases}$

i.e. $X = I(E) \sim \text{Bernoulli}(p)$ with $p = P(E)$

- Binomial Distribution $X \sim \text{Binomial}(n, p)$

$X \doteq$ # of successes in n indep Bernoulli trials with prob. of success p

eg # of heads in n indep. coin flips.

Prob of success = $P(H) = p$ biased
$= 0.5$ fair

pmf. $P(X=k)$ $k = 0, 1, \ldots, n$

$= P($k successes in n indep Bernoulli trials$)$

(eg. $P($k Heads " " " coin flips$)$)

- How many ways to get k successes in n trials?

$\underline{\quad}_1 \; \underline{\quad}_2 \; \ldots \; \underline{\quad}_n$

$\binom{n}{k}$

each of these ways have k successes & $n-k$ failures

so, each has prob $= \underbrace{p \cdots p}_{k \text{ success}} \underbrace{(1-p) \cdots (1-p)}_{n-k \text{ failures}}$

$= p^k (1-p)^{n-k}$

Therefore, $P(X=k) =$ sum of these prob over $\binom{n}{k}$ cases

$= \binom{n}{k} p^k (1-p)^{n-k}$

$$\Rightarrow f_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, 1, 2, \ldots, n \\ 0 & \text{o.w.} \end{cases}$$

<u>Validity:</u> $\sum_{k=0}^{n} f_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \stackrel{?}{=} 1$

<u>Binomial Expansion:</u>

$(a+b)^n = \binom{n}{0} a^0 b^n + \binom{n}{1} a b^{n-1} + \binom{n}{2} a^2 b^{n-2} + \cdots + \binom{n}{n} a^n b^0$

$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$ is the binomial expansion with $a = p$ $b = 1-p$

$= (p + (1-p))^n$

$= 1^n = 1$ ✓

• $E[X] = \sum_{k=0}^{n} k f_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$

$= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$

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$$k\binom{n}{k} = k\frac{n!}{k!\,(n-k)!} = \frac{n!}{(k-1)!\,(n-k)!} = \frac{n\,(n-1)!}{(k-1)!\,((n-1)-(k-1))!}$$

$$= n\binom{n-1}{k-1}$$

$$\Rightarrow E[X] = n\sum_{k=1}^{n}\underbrace{\binom{n-1}{k-1}p^k(1-p)^{n-k}}$$

replace $k-1$ with i $\Rightarrow k = i+1$
and range of summation $k=1, \longrightarrow n$
$\Rightarrow i=0, \longrightarrow n-1$

$$= n\sum_{i=0}^{n-1}\binom{n-1}{i}p^{i+1}(1-p)^{n-(i+1)}$$

$$= np\underbrace{\sum_{i=0}^{n-1}\underbrace{\binom{n-1}{i}p^i(1-p)^{(n-1)-i}}_{\text{pmf of Binomial}(n-1,p)}}_{=1}$$

$$\Rightarrow \boxed{E[X] = np}$$

Similarly $E[X^2] = \sum_{k=0}^{n}k^2 f_X(k) = \sum_{k=1}^{n}k^2\binom{n}{k}p^k(1-p)^{n-k}$

$$k^2\binom{n}{k} = k^2\frac{n!}{k!\,(n-k)!} = \frac{k\,n!}{(k-1)!\,(n-k)!} = nk\frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= nk\binom{n-1}{k-1}$$

$$\Rightarrow E[X^2] = n\sum_{k=1}^{n}k\binom{n-1}{k-1}p^k(1-p)^{n-k} \qquad k-1 \leftrightarrow i \quad k=i+1$$

$$= n\sum_{i=0}^{n-1}(i+1)\,p^{i+1}(1-p)^{n-(i+1)}$$

$$= np\sum_{i=0}^{n-1}(i+1)\,p\,(1-p)^{(n-1)-i}$$

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$$= np\left[\underbrace{\sum_{i=0}^{n-1} i\binom{n-1}{i}p^i(1-p)^{(n-1)-i}}_{\substack{\text{expected value of}\\ \text{Binomial}(n-1,p)\\ =(n-1)p}} + \underbrace{\sum_{i=0}^{n-1}\binom{n-1}{i}p^i(1-p)^{n-1-i}}_{\substack{\text{sum of Bin}(n-1,p)\\ \text{pmf}\\ =1}}\right]$$

$$= np\left((n-1)p+1\right) = n^2p^2 - np^2 + np$$

So, $\text{Var}(X) = E(X^2) - (E(X))^2$

$$= n^2p^2 - np^2 + np - (np)^2$$

$$= np - np^2 = \underline{np(1-p)}$$

- <u>Geometric Distribution</u>: $X \sim \text{Geometric}(p)$

X = # of trials until the first success (indep Bernoulli)

$\Omega = \{1, 2, 3, \dots\}$

<u>pmf</u>: $f_X(k) = P(X=k)$

$$\underbrace{\underset{1}{\_}\ \ \underset{2}{\_}\ \ \dots}_{k-1 \text{ failures}}\ \ \underset{k}{\_}\ \text{first success}$$

prob $(1-p)(1-p)\cdots(1-p) \times p$

$= (1-p)^{k-1}p$

$= p(1-p)^{k-1}$ for $k = 1, 2, 3, \dots$

<u>Validity</u>: $\sum_{k=1}^{\infty} f_X(k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} \overset{?}{=} 1$

$k-1 \longrightarrow i$ (i.e. $k = i+1$)

$$= p\sum_{k=1}^{\infty}(1-p)^{k-1}$$

$$= p\sum_{i=0}^{\infty}(1-p)^i = p\frac{1}{1-(1-p)} = \frac{p}{p} = 1.$$

where we used the fact that

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \quad \text{for } 0<r<1.$$

- $E[X] = \sum_k k f_X(k) = \sum_{k=1}^{\infty} k\, p(1-p)^{k-1}$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1} = -p \sum_{k=1}^{\infty} \frac{d}{dp}(1-p)^k$$

$$= -p \frac{d}{dp}\left(\sum_{k=1}^{\infty}(1-p)^k\right) = -p\frac{d}{dp}\left(\sum_{k=0}^{\infty}(1-p)^k - 1\right)$$

$$= -p\frac{d}{dp}\left(\frac{1}{1-(1-p)} - 1\right) =$$

$$= -p\frac{d}{dp}\left(\frac{1}{p} - 1\right) = -p\left(\frac{-1}{p^2}\right) = \frac{1}{p}$$

- If $X = \begin{cases} 4 & \text{wp } p \\ 8 & \text{wp } 1-p \end{cases}$ & $U \sim \text{Bernoulli}(p) = \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } 1-p \end{cases}$

$$X = 8 - 4U$$

- $Y = \begin{cases} 8 & \text{wp } p \\ 4 & \text{wp } 1-p \end{cases} \rightarrow Y = 4 + 4U.$

Lecture 3 – Continuous RVs

X: a cts rv

· probability density function (pdf): $f_X(x)$

$$P(X \in (a,b)) = \int_a^b f_X(x)\,dx \quad \text{for } a \le b$$

$$\Rightarrow P(X \in (a,b)) = P(X \in [a,b)) = P(X \in (a,b]) = P(X \in [a,b])$$

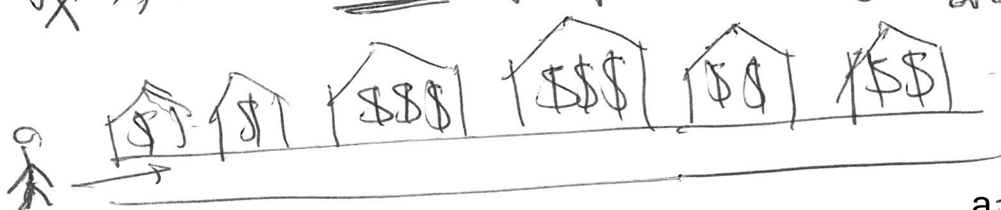
$$\Rightarrow \underbrace{P(X \in [a,a])}_{P(X=a)} = \int_a^a f_X(x)\,dx = 0$$

$\boxed{\text{Caveat: If } X \text{ is discrete, } P(X=i) = f_X(i) \text{ can be } 0 \text{ or can be } > 0.}$

If X is continuous, its cdf is $F_X(x) = P(\underbrace{X \le x}_{X \in (-\infty, x]}) = \int_{-\infty}^{x} f_X(x)\,dx$

$$= P(X < x).$$

· pdf, $f_X(x)$, is the $\underline{\text{rate}}$ of probability accumulating around x



stores & their revenues

$$\frac{P(X \in (a, a+\Delta a))}{\Delta a} = \frac{1}{\Delta a}\int_a^{a+\Delta a} f(x)\,dx \approx \frac{f(a)}{\Delta a}\int_a^{a+\Delta a} dx = \frac{f(a)\Delta a}{\Delta a} = f(a)$$

More precisely

$$\lim_{\Delta a \to 0} \frac{P(X \in (a, a+\Delta a))}{\Delta a} = \lim_{\Delta a \to 0} \frac{F(a+\Delta a) - F(a)}{\Delta a} = F'(a) = f(a).$$

Properties

(I) Validity (i) $f_X(x) \geq 0$ for all x

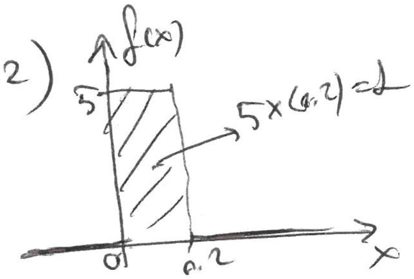
(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Notes: (1) $f_X(x)$ can be > 1 b/c it is not a probability, but rate of change in probability!

If X is discrete
pmf: $f_X(i) = P(X=i)$ is a probability, so
$0 \leq f_X(i) \leq 1$ for discrete X

(2) cdf $F_X(x) = P(X \leq x)$ for X cts or discrete,
so it is a probability " " " "
$\Rightarrow 0 \leq F_X(x) \leq 1$ " " " "

Ex for Note (1) $f_X(x) = \begin{cases} 5 & \text{for } x \in (0, 0.2) \\ 0 & \text{o.w.} \end{cases}$



(II) $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$E[X^i] = \int_{-\infty}^{\infty} x^i f_X(x)\,dx$: i^{th} moment

$Var(X) = E[X^2] - (E(X))^2$

$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

For discrete X
$E[g(X)] = \sum_x g(x) f_X(x)$

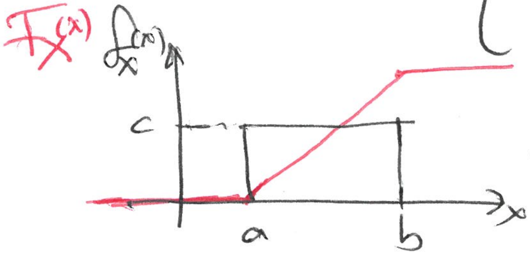
e.g. $E[\sin^2(X)] = \int_{-\infty}^{\infty} \sin^2(x) f_X(x)\,dx$

cdf: $F(x) = \int_{-\infty}^{x} f_X(x)\,dx \Rightarrow F'(x) = \frac{d}{dx} F(x) = f_X(x)$ where F is cts

Uniform Distribution: $X \sim \text{Uniform}(a,b)$, $a<b$ (3)

pdf: $f_X(x) = \begin{cases} c = \dfrac{1}{b-a} & a<x<b \\ 0 & \text{o.w.} \end{cases}$

checks $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = 1.$



$\displaystyle\int_a^b f(x)\,dx = \int_a^b c\,dx = c\int_a^b dx = c(b-a) = 1$

$c = \dfrac{1}{b-a}$

cdf: $\displaystyle F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)\,dx$

for $x<a$, $\displaystyle F_X(x) = \int_{-\infty}^{x} 0\,dx = 0$

" $a \le x < b$, $\displaystyle F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = 0 + \int_a^x \frac{1}{b-a}\,dx$

$\displaystyle = \frac{1}{b-a}\, x\Big|_a^x = \frac{x-a}{b-a}$

" $x \ge b$, $\displaystyle F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = 0 + \int_a^b \frac{1}{b-a}\,dx + 0$

$\displaystyle = \frac{b-a}{b-a} = 1.$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x<a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$$

$\displaystyle E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = 0 + \int_a^b \frac{x}{b-a}\,dx + 0$

$\displaystyle = \frac{1}{b-a}\,\frac{x^2}{2}\Big|_a^b = \frac{1}{2(b-a)}(b^2-a^2)$

$\displaystyle = \frac{a+b}{2}$

Exponential Distribution: $X \sim \text{Exponential}(\lambda)$

$\text{Exp}(\lambda)$

pdf: $$f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x > 0 \\ 0 & \text{o.w.} \end{cases} \qquad \lambda > 0$$

- used to model lifetime of electronic components or waiting times between calls

Validity: $$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^{\infty} \frac{1}{\lambda} e^{-x/\lambda} dx = \frac{1}{\lambda} (-\lambda) e^{-x/\lambda} \Big|_0^{\infty}$$

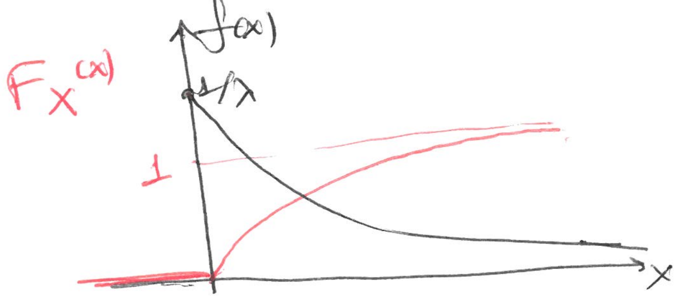
$$= -(0 - 1) = 1 \checkmark$$

- $E[X] = \int_0^{\infty} x \frac{1}{\lambda} e^{-x/\lambda} dx = \ldots = \lambda$

  use integration by parts with $u = x$ & $dv = e^{-x/\lambda} dx$

  $= uv - \int v\, du \ldots$

- $E[X^2] = \int_0^{\infty} x^2 \frac{1}{\lambda} e^{-x/\lambda} dx = \ldots = 2\lambda^2$ use integration by parts twice



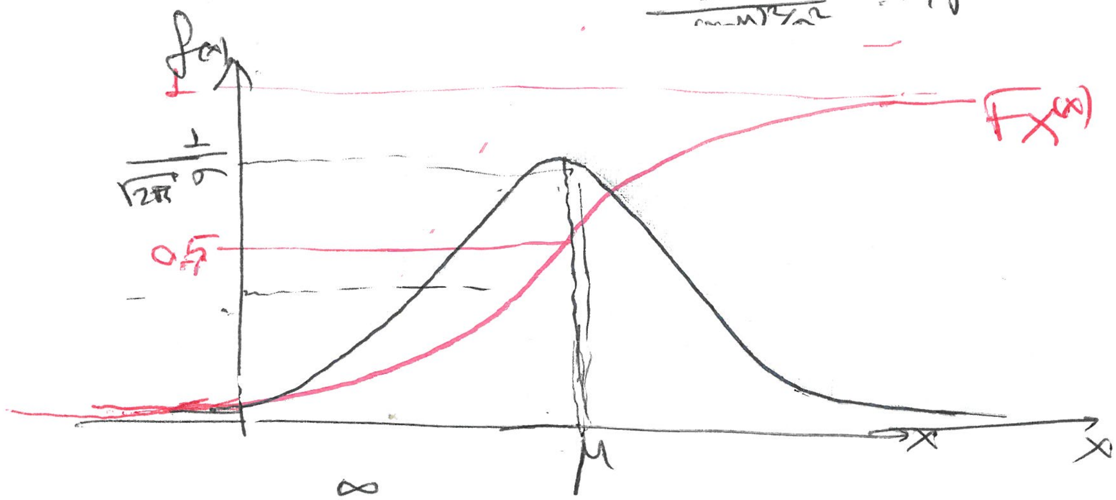
$$\Rightarrow Var(X) = E(X^2) - E(X)^2$$
$$= 2\lambda^2 - \lambda^2$$
$$= \lambda^2.$$

- **Normal Distribution** : $X \sim N(\mu, \sigma^2)$

pdf: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}$

$\uparrow$

$\frac{1}{e^{(x-\mu)^2/\sigma^2}}$: exponential decay



Validity: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ (will show this for standard normal distr.)

- $E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \dots = \mu$

let $u = \frac{x-\mu}{\sigma}$ ..

- $Var(X) = \sigma^2 \qquad SD(X) = \sigma$.

- If $\mu = 0$, $\sigma = 1$, we have standard normal dist.

$Z \sim N(0,1)$

pdf: $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

- cdf: no closed form.
- Validity: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1$, which follows from

$$\int_0^\infty e^{-x^2/2}\,dx = \sqrt{\pi/2}$$

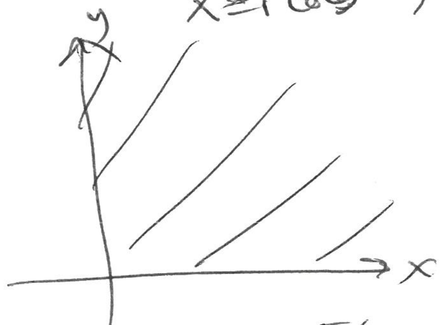
__Sol^n:__ Let $I = \int_0^\infty e^{-x^2/2}dx$

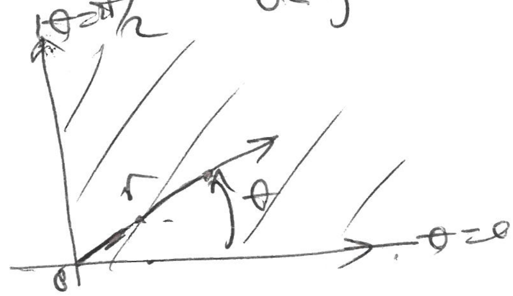
$$I \times I = \left(\int_0^\infty e^{-x^2/2}dx\right)\left(\int_0^\infty e^{-y^2/2}dy\right)$$

$$\Rightarrow I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)/2}\,dx\,dy$$

Switching to polar coordinates,

$x = r\cos\theta$, $y = r\sin\theta$ $\quad$ $0<x<\infty$, $0<y<\infty$ $\Rightarrow$ $r>0$, $0<\theta<\frac{\pi}{2}$





$$\Rightarrow I^2 = \int_0^\infty\int_0^{\pi/2} e^{-r^2/2}\, r\,d\theta\,dr$$

$$= \frac{\pi}{2}\int_0^\infty r e^{-r^2/2}dr = \frac{\pi}{2}\left(-e^{-r^2/2}\right)\Big|_0^\infty = -\frac{\pi}{2}(0-1)$$

$$= \frac{\pi}{2}$$

$$\Rightarrow I = \sqrt{\frac{\pi}{2}}$$

So $\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}} = \frac{1}{2}$

So $\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx = \frac{1}{2}+\frac{1}{2}$

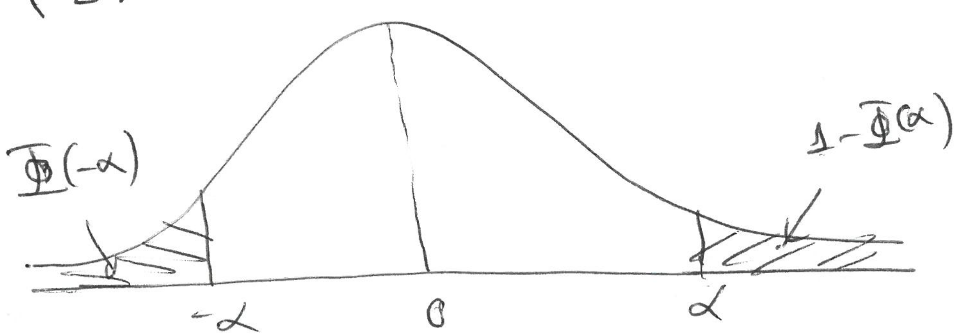
$$= \underline{\underline{1}}$$

- For $Z \sim N(0,1)$, $E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0$

cdf: $F_Z(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = P(Z \le \alpha) = \Phi(\alpha)$

Phi of α

- $P(Z > \alpha) = 1 - P(Z \le \alpha) = 1 - \Phi(\alpha)$



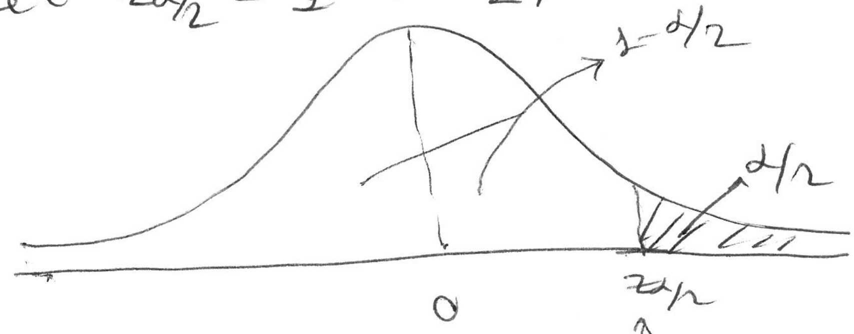
By symmetry, $\Phi(-\alpha) = 1 - \Phi(\alpha)$

<u>Ex:</u> $P(|Z| > \alpha) = P(Z > \alpha) + P(Z < -\alpha)$

$= 1 - \Phi(\alpha) + \Phi(-\alpha)$

$= 2(1 - \Phi(\alpha)) = 2\Phi(-\alpha)$

- Let $z_{\alpha/2} = \Phi^{-1}(1 - \frac{\alpha}{2})$



<u>Qu:</u> Where does $z_{\alpha/2}$ lie on the above plot?

$\Phi(z_{\alpha/2}) = 1 - \alpha/2$

$P(Z \le z_{\alpha/2}) = 1 - \frac{\alpha}{2}$

for $0 < \alpha < 1$

Transformation Property (or Standardization)

- If $X \sim N(\mu, \sigma^2)$ & $Y = aX + b$ then $(*)$

$$Y \sim N(a\mu + b, a^2\sigma^2)$$

- Note: This does not hold in general:

Ex: $X \sim Exp(\lambda)$, $Y = X + 2$ Is $Y \sim Exp(\ )$?

- Special Case:

If $X \sim N(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

Exer: ~~Hint:~~ take $a = \frac{1}{\sigma}$ & $b = \frac{-\mu}{\sigma}$ in $(*)$ above

- This means that to compute probabilities for X, you need to know probabilities for $Z = \frac{X-\mu}{\sigma}$, i.e. values of Φ.

Ex: $X \sim N(10, 25)$, then $P(X \leq 20) = ?$

$\Rightarrow \mu = 10, \sigma = 5$

$$P(X \leq 20) = P\left(\frac{X - 10}{5} \leq \frac{20 - 10}{5}\right) =$$

$$= P(Z \leq 2) = \Phi(2) = \ldots$$

Ex: For $X \sim N(10, 25)$ find x_0, s.t. $P(X > x_0) = 0.80$

$$\Rightarrow P\left(\frac{X - 10}{5} > \frac{x_0 - 10}{5}\right) = P(Z > \underbrace{\frac{x_0 - 10}{5}}_{z_0}) = 0.80$$

$$1 - P(Z < z_0) = 1 - \Phi(z_0) = 0.80$$

$$\Rightarrow \Phi(z_0) = 0.20 \qquad z_0 = \Phi^{-1}(0.20)$$

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Distribution	pdf	Conditions	E(X)	Var(X)
(1) Unif(a,b)	$\frac{1}{b-a}$	$a<x<b$ $a<b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
(2) Exp(λ)	$\frac{1}{\lambda}e^{-x/\lambda}$	$x>0$ $\lambda>0$	λ	λ^2
(3) $N(\mu,\sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$x\in\mathbb{R}$ $\mu\in\mathbb{R}$ $\sigma>0$	μ	σ^2