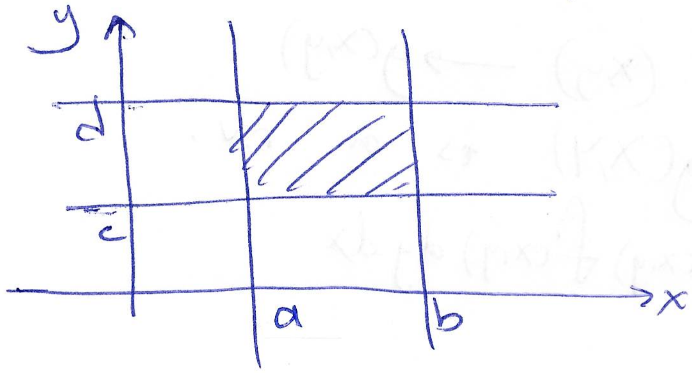
Lecture 4 - Joint Distributions

Snow = g(temperature, humidity)

Ex: $P(\text{Snow} > 1\text{inch}) = P(\text{temp} \le 30°F \cap \text{humid} > 30\%)$

In general, $P(X \in (a,b) \cap Y \in (c,d))$

$= P(X \in (a,b), Y \in (c,d))$



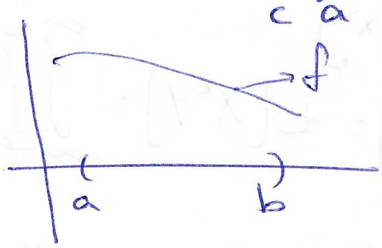
$$\int_a^b \int_c^d \underbrace{f_{X,Y}(x,y)}_{\text{joint pdf}}\, dy\, dx$$

$$= \int_c^d \int_a^b f_{X,Y}(x,y)\, dx\, dy$$

Recall Single rv case

$P(X \in (a,b))$

$= \int_a^b f(t)\, dt$



Joint cdf $F_{X,Y}(\alpha,\beta) = P(X \le \alpha, Y \le \beta)$

$$= \int_{-\infty}^{\alpha} \left(\int_{-\infty}^{\beta} f(x,y)\, dy \right) dx$$

$F(-\infty,-\infty) = 0$

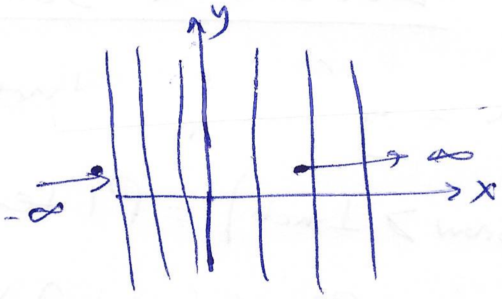
$F(\infty,\infty) = 1$

Validity: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y} = f_{X,Y}(x,y)$$

Marginal distributions

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = f_Y(y)$$



$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = f_X(x)$$

Expectation: (Joint)

$(x,y) \longrightarrow g(x,y)$

$g(X,Y)$ is a r.v.

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dy\, dx$$

Ex: $g(x,y) = x^2 y$

$$E[g(X,Y)] = E[X^2 Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 y\, f(x,y)\, dy\, dx$$

Ex: $g(x,y) = x$

$$E[g(X,Y)] = E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f(x,y)\, dy \right) dx = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Linearity of Expectation (LOE)

$$E[X+Y] = E[X] + E[Y]$$

$$E[X+Y+Z] = E[X] + E[Y] + E[Z]$$

$$E[g(X) + h(Y)] = E[g(X)] + E[h(Y)]$$

Ex: $E[\ln^2(X) + \sin^3(Y)] = E[\ln^2(X)] + E[\sin^3(Y)]$

Pf (of LOE)

(i) continuous case

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dy\,dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\left(\int_{-\infty}^{\infty} f(x,y)\,dy\right)}_{f_X(x)} dx + \int_{-\infty}^{\infty} y \underbrace{\left(\int_{-\infty}^{\infty} f(x,y)\,dx\right)}_{f_Y(y)} dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\,dx + \int_{-\infty}^{\infty} y\, f_Y(y)\,dy$$

$$= E[X] + E[Y]$$

(ii) Discrete case is similar with replacing integrals with sums.

Note: For LOE, X & Y need not to be independent!

Extension: $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$

Ex: $X \sim \text{Binomial}(n,p)$ Show that $E[X]=np$

Sol'n: X = # of successes (i.e. Heads) in n flips of a coin with $P(H)=p$

$\Rightarrow X = \sum_{i=1}^{n} X_i$ where

$$X_i = I_{F_i} = \begin{cases} 1 & \text{if } i^{th} \text{ flip is heads (or success)} \\ 0 & \text{" " " " tails (or failure)} \end{cases} \quad \text{for all } i=1,\dots,n$$

so, $X_i \sim \text{Bernoulli}(p)$ with pmf $f(x) = \begin{cases} p & \text{if } x=1 \\ 1-p & \text{if } x=0 \end{cases}$

$\Rightarrow E[X_i] = E[I_{F_i}] = P(F_i) = p$

Then $E[X] = E\left[\sum_{i=1}^{n} X_i\right]$

$$= \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np.$$

Note: $X_i = I_{F_i}$ where $F_i = i^{th}$ flip is success will give the same result.

Ex: (The Hat Problem)

n people throw their hats in a box and then each pick one at random.

X: # of people who get their own hat.

$E[X] = ?$

Sol'n:

$$X_i = I_{H_i} = \begin{cases} 1 & \text{if } i^{th} \text{ person gets his/her hat back} \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow X = \sum_{i=1}^{n} X_i = I_{H_1} + I_{H_2} + \dots + I_{H_n}$$

$$\Rightarrow E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P(H_i) = \sum_{i=1}^{n} \frac{1}{n} = n\frac{1}{n} = 1$$

Note: Pf of $P(H_i) = \frac{1}{n}$ for all i

$P(H_1) = \frac{1}{n}$

$P(H_2) = P(H_2 \mid \text{person 1 got hat 2}).\, P(P_1 \text{ got hat 2})$

$\quad + P(H_2 \mid$ " " didnot get hat2$)\, P(P_1 \text{ not get hat2})$

$$= 0 \cdot \frac{1}{n} + \frac{1}{n-1} \cdot \frac{n-1}{n} = \frac{1}{n}$$

$\vdots$

Independence of RV's

Recall events $A \perp B$ iff $P(A \cap B) = P(A)P(B)$

(i) discrete case: $X \perp Y$ iff $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

(ii) cts ": $X \perp Y$ iff $F_{X,Y}(x,y) = F_X(x) F_Y(y)$.

$\Leftrightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Product of Expectation (POE)

If $X \perp Y$ then $E[XY] = E[X]E[Y]$

Pf: $E[\underbrace{XY}_{g(x,y)=xy}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dy\, dx$ } indep of X & Y

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x f_X(x) \Big(\underbrace{\int_{-\infty}^{\infty} y f_Y(y)\, dy}_{E[Y]} \Big)\, dx$$

$$= E[Y] \underbrace{\int_{-\infty}^{\infty} x f_X(x)\, dx}_{E[X]}$$

$$= E[X]E[Y] \quad \blacksquare$$

Exer: Do the pf for discrete X & Y.

Extension (of POE)

If $X \perp Y$ then $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.

Linearity of Variance (LOV) - for indep r.v.s

If $X \perp Y$, then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$.

Pf: $\text{Var}(X+Y) = E\left[\left((X+Y) - E(X+Y)\right)^2\right]$

$$= E\left[\left((X-E(X)) + (Y-E(Y))\right)^2\right]$$

$$= E\left[(X-E(X))^2\right] + E\left[(Y-E(Y))^2\right] + E\left[2(X-E(X))(Y-E(Y))\right]$$

$$= \text{Var}(X) + \text{Var}(Y) + 2E\left[\underbrace{(X-E(X))}_{g(X)}\underbrace{(Y-E(Y))}_{h(Y)}\right]$$

where $E\left[(X-E(X))(Y-E(Y))\right]$

$$= E[X-E(X)]\,E[Y-E(Y)]$$

$$= (E(X)-E(X))(E(Y)-E(Y)) = 0 \cdot 0 = 0.$$

$\Rightarrow \text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ $\boxed{8}$

Ex: $X \sim \text{Binomial}(n,p)$.

$X = \sum_{i=1}^{n} X_i$, $\quad X_i = I_{F_i} = \begin{cases} 1 & \text{if flip } i \text{ is success} \\ 0 & \text{" " " " failure} \end{cases}$

& X_is are indep.

$\Rightarrow E[X] = E\left(\sum X_i\right) \overset{LOE}{=} \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} P(F_i) = \sum_{i=1}^{n} p = np$

$\text{Var}(X) = \text{Var}\left(\sum X_i\right) \overset{LOV}{=} \sum \text{Var}(X_i) = \sum_{i=1}^{n} p(1-p) = np(1-p)$.

Ex: $X \sim F_X$ & $Y \sim F_Y$ & $X \perp Y$.

(a) Let $M = \max(X,Y)$. $F_M(\alpha) = ?$

$$F_M(\alpha) = P(M \le \alpha) = P(\max(X,Y) \le \alpha)$$
$$= P(X \le \alpha, Y \le \alpha)$$
$$= P(X \le \alpha)\, P(Y \le \alpha) \quad \} X \perp Y$$
$$= F_X(\alpha)\, F_Y(\alpha).$$

(b) Let $W = \min(X,Y)$. $F_W(\alpha) = ?$

$$F_W(\alpha) = P(W \le \alpha)$$
$$= P(\min(X,Y) \le \alpha)$$
$$= 1 - P(\min(X,Y) > \alpha)$$
$$= 1 - P(X > \alpha, Y > \alpha)$$
$$= 1 - P(X > \alpha)\, P(Y > \alpha) \quad \} X \perp Y$$
$$= 1 - (1 - F_X(\alpha))(1 - F_Y(\alpha))$$

Normal Distribution

Linear Transformation: If $X \sim N(\mu, \sigma^2)$ then for $a, b \in \mathbb{R}$, $Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$

Weighted Sum (i.e. Linear Combination) of $\perp$ Normals

(i) If $X_1 \sim N(\mu_1, \sigma_1^2) \perp X_2 \sim N(\mu_2, \sigma_2^2)$
then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

(ii) If $X_i \sim N(\mu_i, \sigma_i^2)$ & X_i's are $\perp$ then
$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$ where a_i's are constant

Ex: $X \sim N(2,4) \perp Y \sim N(1,5)$

(1) $X+Y \sim N(2+1, 4+5) \equiv N(3,9)$

(2) $X+3Y \sim N(2+3\times 1, 4+9\times 5)$
$= N(5, 49)$

(3) $X-Y = X+(-Y) \sim N(1\cdot 2+(-1)1, 1^2\cdot 4+(-1)^2 5)$
$= \underbrace{1}_{a_1}X + \underbrace{(-1)}_{a_2}Y \qquad = N(1,9)$

iid $\equiv$ independent identically distributed

"X & Y are iid rv's" $\Leftrightarrow$ (i) $X \perp Y$; (ii) $X \overset{d}{=} Y$ $\rightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y)$
$= f_X(x) f_X(y)$

Ex: $X \sim$ Uniform $(0,1)$, $Y \sim$ Uniform $(0,2)$, $W \sim$ Uniform $(0,1)$ $\rightarrow X \overset{d}{=} W \Rightarrow$ $E(X) = E(W)$, $E(X^2) = E(W^2)$, $Var(X) = Var(W)$, $F_X(\alpha) = F_W(\alpha)$ ...

$\boxed{\text{Note: If } X=Y \text{ then } X \overset{d}{=} Y \text{ but not vice versa.}}$

Ex:

(i) $X_1 = X_2$	(ii) $X_1 \overset{d}{=} X_2$
$E[X_1+X_2] = E[X_1+X_1]$ $= E[2X_1]$ $= 2E[X_1]$	$E[X_1+X_2] \overset{LOE}{=} E[X_1] + E[X_2]$ $= E[X_1] + E[X_1]$ $= 2E[X_1]$
$E[(X_1+X_2)^2] = E[(2X_1)^2]$ $= E[4X_1^2]$ $= 4E[X_1^2]$	$E[(X_1+X_2)^2] = E[X_1^2+2X_1X_2+X_2^2]$ $= E[X_1^2] + 2E[X_1X_2] + E[X_2^2]$ $= 2E[X_1^2] + 2E[X_1X_2]$

Ex (iii) X_1 & X_2 are iid rv's. (i.e. $X_1 \overset{iid}{\sim} X_2$)

Then $E[(X_1+X_2)^2] = 2E[X_1^2] + 2E[X_1X_2]$ (from identical distr.)

$= 2E[X_1^2] + 2E[X_1]E[X_2]$ } $\perp$

$= 2E[X_1^2] + 2(E[X_1])^2$

Exer: Find $Var(X_1+X_2)$ in cases (i)-(iii) above

Ex: X & Y are iid rv's with $E(X)=\mu$ & $Var(X)=\sigma^2$

Find $E[X^2+XY+Y^2]$ in terms of μ & σ^2.

Sol'n: $E[X^2+XY+Y^2] \overset{LOE}{=} E[X^2] + E[XY] + E[Y^2]$

$= 2E[X^2] + E(X)E(Y)$

$= 2E(X^2) + (E(X))^2$

Recall $Var(X) = E(X^2) - (E(X))^2 \Rightarrow \sigma^2 = E(X^2) - \mu^2$

$\Rightarrow E(X^2) = \mu^2 + \sigma^2$

So, $E[X^2+XY+Y^2] = 2(\mu^2+\sigma^2) + \mu^2 = 3\mu^2 + 2\sigma^2$

Ex: What if $X=Y$ in above Ex?

$E[X^2+XY+Y^2] = E[X^2+XX+X^2] = E[3X^2]$

$= 3E[X^2] = 3(\mu^2+\sigma^2)$

Recap

1. LOE: $E(\sum X_i) = \sum E[X_i]$ $\quad\perp$ not needed

   or $E(\sum a_i X_i) = \sum a_i E[X_i]$

2. POE: If $X \perp Y$, then

   $$E(XY) = E(X)E(Y)$$

   also $E(g(X)h(Y)) = E(g(X))E(h(Y))$

3. LOV: If $X \perp Y$, then

   $$Var(X+Y) = Var(X) + Var(Y)$$

   also $Var(g(X)+h(Y)) = Var(g(X)) + Var(h(Y))$

Let X be any r.v. & c be a constant.

Then

- $E[c] = c$, $E[c^2] = c^2 \Rightarrow Var(c) = E[c^2] - (E[c])^2 = c^2 - c^2 = 0$
- $E[cX] = cE[X]$, $E[(cX)^2] = E[c^2X^2] = c^2 E[X^2]$

  $$\Rightarrow Var(cX) = E[(cX)^2] - (E[cX])^2 = c^2E[X^2] - (cE[X])^2$$
  $$= c^2E[X^2] - c^2(E[X])^2$$
  $$= c^2(E[X^2] - (E[X])^2)$$
  $$= c^2 Var X$$

Let X & Y be two r.v's & $X \perp Y$, c a constant

Then

- $Var(X+cY) = Var(X) + c^2 Var(Y)$

  $Var(\underbrace{X-cY}_{X+(-c)Y}) = Var(X) + c^2 Var(Y)$

Recall

$$E[X_1+X_2] = 2E[X_1] \quad \text{for all cases of } X_1 = X_2,\ X_1 \overset{d}{=} X_2 \text{ and } X_1 \overset{iid}{\sim} X_2$$

Consider $X_1 - X_2$ in these cases

(i) $X_1 = X_2$

- $E[X_1 - X_2] = E[X_1 - X_1] = E[0] = 0$
- $E[(X_1-X_2)^2] = E[(X_1-X_1)^2] = E[0] = 0$

(ii) $X_1 \overset{d}{=} X_2$

- $E[X_1 - X_2] = E[X_1] - E[X_2] = E[X_1] - E[X_1] = 0$
- $E[(X_1-X_2)^2] = E[X_1^2 - 2X_1X_2 + X_2^2]$

$$= \underline{E[X_1^2]} - 2E[X_1X_2] + \underline{E[X_2^2]}$$

$$= 2E[X_1^2] - 2E[X_1X_2]$$

(iii) $X_1 \overset{iid}{\sim} X_2$

- $E[X_1 - X_2] = 0$
- $E[(X_1-X_2)^2] = 2E[X_1^2] - 2E[X_1X_2]$

$$= 2E[X_1^2] - 2\underline{E[X_1]}\,\underline{E[X_2]}$$

$$= 2E[X_1^2] - 2(E[X_1])^2$$

<u>Exer:</u> Find $Var(X_1 - X_2)$ in cases (i)-(iii) above