Lecture 9 - Stochastic Processes & Markov Chains

Stochastic Process (SP): A SP in discrete time $t \in \{0,1,2,\dots\}$ is a sequence of time-indexed RV's $\underbrace{X_0}_{\text{RV at time}=0}, \underbrace{X_1}_{\text{RV at time } t=1}, \dots$ etc. with $\mathcal{X} = \{X_t, t \geq 0\}$

- If t is continuous then cts SP
- " " " discrete " discrete " .

Markov Chains (MC) → Discrete Time MC (DTMC)
→ Cts " MC (CTMC)

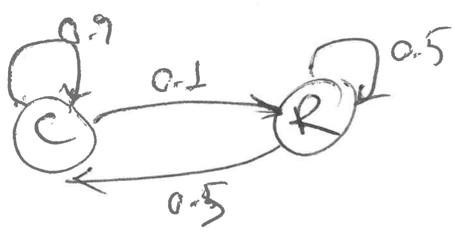
Markovian Property: The transition to the next state depends only on the current state.

Ex: Tracking Daily weather $\mathcal{X} = \{\underbrace{\text{Clear}}_{C}, \underbrace{\text{Rainy}}_{R}\}$

today $C \xrightarrow{0.9} C$ tomorrow $\quad R \xrightarrow{0.5} R$

$C \xrightarrow{0.1} R \quad R \xrightarrow{0.5} C$

Assumption: a day's weather depends only on the previous day



Markov Chain

Distribution of escaping a state is Geo(p).?

A Stochastic Process with Markovian property is called a <u>Markov Chain</u>.

<u>Discrete Time Markov Chain (DTMC)</u>: A SP $\mathcal{X} = \{X_t, t \geq 0, t \text{ discrete}\}$ is a <u>DTMC</u> if for all t

$$\begin{cases} P(X_{t+1} = j \mid X_t = i, X_{t-1} = a, X_{t-2} = b, \ldots) = \\ P(X_{t+1} = j \mid X_t = i) = P_{ij} \end{cases}$$

$\leftarrow$ one-step transition probability (from i to j)

Markovian Property: (recall) A transition to state $t+1$, X_{t+1}, only depends on the current state X_t.

$$P_{ij} = P(\underset{\text{eg. today}}{i} \longrightarrow \underset{\text{tomorrow}}{j})$$

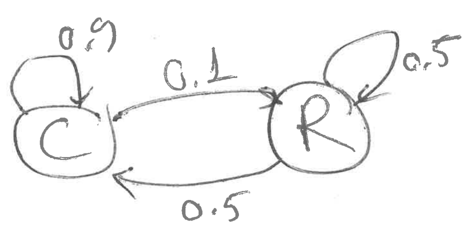
<u>Ex</u>: Recall daily weather $\mathcal{X} = \{\overset{\text{clear}}{C}, \overset{\text{rainy}}{R}\}$

X_1: weather on day 1 $\in \mathcal{X}$

X_2: " " " 2 $\in \mathcal{X}$

X_3: " " " 3 $\in \mathcal{X}$

$\vdots$



eg. $\underset{\text{clear today}}{C} \xrightarrow{0.9} \underset{\text{clear tomorrow}}{C}$

$P_{CC} = 0.9$ $\quad$ $P_{CR} = 0.1$

$P_{RC} = 0.5$ $\quad$ $P_{RR} = 0.5$ $\quad \leftarrow P_{ij} = P(i \to j)$

$$P = \begin{array}{c c} & \overbrace{\begin{array}{cc} C & \quad R \end{array}}^{\text{states tomorrow}} \\ \underbrace{\begin{array}{c} C \\ R \end{array}}_{\text{states today}} & \begin{bmatrix} P_{CC} & P_{CR} \\ P_{RC} & P_{RR} \end{bmatrix} \end{array} = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \leftarrow \text{transition (prob.) matrix}$$

$P_{ij} = P(\text{row } i \to \text{col } j)$
$= P(i \to j)$

rows of P sum to 1:

$$P_{CC} + P_{CR} = 0.9 + 0.1 = 1$$
$$P_{RC} + P_{RR} = 0.5 + 0.5 = 1$$

In general $\sum_j P_{ij} = 1 \rightarrow \sum_j P(i \to j) = P(i \to \text{any state})$

(P_{ij}^2) : Prob ($i \to j$ in 2 steps)

$$(P_{ij}^2) = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \times \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} (P_{00}^2) & (P_{01}^2) \\ (P_{10}^2) & (P_{11}^2) \end{bmatrix} = P^2$$

e.g. $(P_{00}^2) = \underline{P_{00} \cdot P_{00}} + \underline{P_{01}}\, \underline{P_{10}}$

In general (P_{ij}^n) = Prob($i \to j$ in n steps)
(path does not matter)
$= P^n \leftarrow$ multiply P with itself n times.

If there are $k+1$ states; $0, 1, \dots, k$

$$P^n = \begin{bmatrix} (P_{00}^n) & (P_{01}^n) & \cdots & (P_{0k}^n) \\ \vdots & \vdots & \ddots & \vdots \\ (P_{k0}^n) & (P_{k1}^n) & \cdots & (P_{kk}^n) \end{bmatrix}$$

Ex: $P^2 = \begin{bmatrix} 0.86 & 0.14 \\ 0.70 & 0.30 \end{bmatrix}$, $\quad P^5 = \begin{bmatrix} 0.83504 & 0.16496 \\ 0.82480 & 0.17520 \end{bmatrix}$

$$P^{10} = \begin{bmatrix} 0.8333508 & 0.1666492 \\ 0.8332460 & 0.1667540 \end{bmatrix} \quad \cdots \quad P^{100} = \begin{bmatrix} 0.8\bar{3} & 0.1\bar{6} \\ 0.8\bar{3} & 0.1\bar{6} \end{bmatrix}$$

Ex: $\mathcal{X} = \{C, S, R\}$ — C: clear, S: snowy, R: rainy

$C \xrightarrow{0.7} C \qquad S \xrightarrow{0.6} C \qquad R \xrightarrow{0.4} C$

$C \xrightarrow{0.2} S \qquad S \xrightarrow{0.3} S \qquad R \xrightarrow{0.4} S$

$C \xrightarrow{0.1} R \qquad S \xrightarrow{0.1} R \qquad R \xrightarrow{0.2} R$

$$P = \begin{array}{c} \\ C \\ S \\ R \end{array} \begin{array}{c} \begin{array}{ccc} C & S & R \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.6 & 0.3 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{bmatrix} \end{array} \longrightarrow P^5 = \begin{bmatrix} 0.64195 & 0.24690 & 0.111111 \\ 0.64198 & 0.24691 & 0.111111 \\ 0.64188 & 0.24700 & 0.11112 \end{bmatrix}$$

$$\longrightarrow P^{10} = \begin{bmatrix} 0.6419753 & 0.2469136 & 0.\bar{1} \\ '' & '' & '' \\ '' & '' & '' \end{bmatrix}$$

In general, for large n,

$$P^n = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_k \\ \pi_1 & \pi_2 & \cdots & \pi_k \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_k \end{bmatrix} \begin{array}{l} \text{sum} = 1 \\ \text{sum} = 1 \\ \\ \text{sum} = 1 \end{array}$$

Each row is the same

i.e. Prob of ending up at state j after sufficiently large # of steps does not depend on the initial state or first few states

$$\lim_{n \to \infty} (P_{ij}^n) = \pi_j \leftarrow \text{does not depend on } i$$

$\pi_j \geq 0$ &

$\sum_j \pi_j = 1$, so $\vec{\pi} = (\pi_1, \ldots, \pi_k)$ is a pmf.

$\pi_j = \lim_{n\to\infty} (P^n_{ij}) = \lim_{n\to\infty} P(i \longrightarrow j \text{ in } n \text{ steps})$

= long term prob. of being in state j

= <u>limiting</u> prob of being in state j

$\vec{\pi}$ is called <u>limiting distr.</u> or <u>stationary distr.</u>

<u>Goal:</u> How to find $\vec{\pi}$?

<u>Application:</u> mean or average state

$= \sum_{i=1}^{k} i\pi_i$ (in the long run)

Recall that for large n — each row is vector $\vec{\pi}$

$$P^{n+1} = P^n = \begin{pmatrix} \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \\ \vdots \\ \vec{\pi} \longrightarrow \end{pmatrix}$$

$$\Rightarrow \quad P^n P = \begin{pmatrix} \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \\ \vdots \\ \vec{\pi} \longrightarrow \end{pmatrix} P = \begin{pmatrix} \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \\ \vdots \\ \vec{\pi} \longrightarrow \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\vec{\pi} P = \vec{\pi}}$$

<u>Ex:</u> $\mathcal{X} = \{C, R\}$ $\quad \vec{\pi} = (\pi_C, \pi_R)$

$$(\pi_C \;\; \pi_R) \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (\pi_C \;\; \pi_R)$$

$$\Rightarrow \left.\begin{aligned} 0.9\pi_C + 0.5\pi_R &= \pi_C \\ 0.1\pi_C + 0.5\pi_R &= \pi_R \end{aligned}\right\} \xrightarrow{\text{both eqns yield this}} \Rightarrow 0.5\pi_R = 0.1\pi_C$$

$$\Rightarrow 5\pi_R = \pi_C$$

$\&\ \pi_C + \pi_R = 1$ (since $\vec{\pi}$ is a pmf)

$$\Rightarrow 5\pi_R + \pi_R = 1 \Rightarrow \pi_R = 1/6$$

$$\rightarrow \pi_C = 5(1/6) = 5/6$$

$$\Rightarrow \vec{\pi} = (5/6 \quad 1/6)$$

Recap: $\vec{\pi} P = \vec{\pi}$ The distribution $\vec{\pi}$ that solves this equation is called the stationary distr. of that M.C.

Thm: If limiting distr. exists, it is also the unique stationary distr.

Ex: $\pi_R = 1/6$ P(rains on day 400) $= 1/6$
P(" " " 401) $= 1/6$

but $P(R \text{ on } 401 \mid R \text{ on } 400) = P_{RR} = 0.5$

check: $P(R \text{ on } 401) = P(R \text{ on } 401 \mid R \text{ on } 400)P(R \text{ on } 400)$
$+ P(R \text{ on } 401 \mid C \text{ on } 400)P(C \text{ on } 400)$
$= 0.5(1/6) + 0.1(5/6) = 2 \times \frac{0.5}{6}$
$= 1/6$

Ex: Find $\vec{\pi}$ for $\mathcal{X} = \{C, S, R\}$ example.
(left as exercise)

Ex:

$$R \xrightarrow{0.5} RR \xrightarrow{0.5} RRR$$

day: 1   1 2   1 2 3

$$R \xrightarrow{0.5} RC \xrightarrow{0.1} RCR$$

prob only depends on the previous state (ordering)

Ex: Umbrella Problem: Prof X commutes between Home & office.

Suppose Prof X has 2 umbrellas.

If it rains, he takes one with him (if available)

prob (rain) $= p$ & prob (no rain) $= 1-p$

Let $i = 0, 1, 2$ be the # of umbrellas he has @ home or @ office in a given day.

Qu: (i) Prob of he gets wet? in the long run (ii) long term prob of ending up with zero Umbrella

# of Umbrellas @ office

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1-p & p \\ 2 & 1-p & p & 0 \end{array}$$

# of Umbrellas @ home

$$\vec{\pi} = (\pi_0, \pi_1, \pi_2)$$

$$\vec{\pi} P = \vec{\pi}$$

$$\Rightarrow (\pi_0 \ \ \pi_1 \ \ \pi_2) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{pmatrix} = (\pi_0 \ \ \pi_1 \ \ \pi_2)$$

$$\Rightarrow \pi_2(1-p) = \pi_0$$

$$(1-p)\pi_1 + p\pi_2 = \pi_1 \rightarrow \pi_1 = \pi_2$$

$$\pi_0 + p\pi_1 = \pi_2$$

$$\rightarrow \pi_2(1-p) + p\pi_2 = \pi_2$$

$$\rightarrow \pi_2 = \pi_1 = \frac{1}{3-p}$$

(ii) $\pi_0 = \dfrac{1-p}{3-p}$

(i) $P(\text{wet}) = \pi_0 \cdot p = \dfrac{(1-p)p}{3-p}$.

Random Walk on integers starting at 0

... -2 $\quad -1$ $\quad 0$ $\quad 1$ $\quad 2$ ...

$$\text{Prob}(\text{left}) = \text{Prob}(\text{right}) = \frac{1}{2}$$