Introduction

Variance

As in Section on Expectation (Ch 3 in Matloff), the concepts and properties introduced in this section form the very core of probability and statistics.

Note: Except for some specific calculations, these apply to both discrete and continuous random variables.

Definition

While the expected value tells us the long-run average of a random variable, we also need a measure of its variability.

In other words, we want a measure of **dispersion** or **spread**. The classical measure of spread is **variance**.

Variance Definition: For a random variable U for which the expected values exist, the **variance** of U is defined to be:

$$Var(U) = E[(U - EU)^2] \tag{1}$$

Die Example

For X in the die example:

$$Var(X) = E[(X - 3.5)^2] \tag{2}$$

Here, $W = (X - 3.5)^2$ is a function of X. We find the expected value of this new random variable W.

Notebook View

line	X	W
1	2	2.25
2	5	2.25
3	6	6.25
4	3	0.25
5	5	2.25
6	1	6.25

Calculating Variance

To evaluate, apply:

$$Var(X) = \sum_{c=1}^{6} (c - 3.5)^2 \cdot \frac{1}{6} = 2.92$$

Variance gives us a measure of dispersion.

Intuition

In the expression $Var(U) = E[(U - EU)^2]$:

- If values of U are clustered near its mean, then $(U - EU)^2$ will usually be small.
- Then variance of U will be small.
- If U varies widely, then its variance will be large.

Properties of Variance

Property F

$$Var(U) = E(U^2) - (EU)^2 \tag{3}$$

The term $E(U^2)$ is evaluated using Property D of expectation.

Derivation of Property F

Using properties of expected values:

$$\begin{aligned} Var(U) &= E[(U - EU)^2] \\ &= E[U^2 - 2EU \cdot U + (EU)^2] \text{ (algebra)} \\ &= E(U^2) + E(-2EU \cdot U) + E[(EU)^2] \text{ (property B)} \\ &= E(U^2) - 2EU \cdot EU + (EU)^2 \text{ (properties C)} \\ &= E(U^2) - (EU)^2 \end{aligned}$$

Note on Property F: Remember, (3) is a shortcut formula for finding the variance, not the *definition* of variance.

Example using Property F

If X is the number of dots which come up when we roll a die:

$$Var(X) = E(X^2) - (EX)^2 \tag{4}$$

From Property D of expectation

$$E(X^2) = \sum_{i=1}^{6} i^2 \cdot \frac{1}{6} = \frac{91}{6} \tag{5}$$

Thus,

$$Var(X) = E(X^2) - (EX)^2 = \frac{91}{6} - 3.5^2 \approx 2.92$$

Property G

An important behavior of variance is:

$$Var(cU) = c^2 Var(U) \tag{6}$$

This means: If we multiply a random variable by c, its variance multiplies by c^2.

Proof of Property G: Defining $V = cU$:

$$\begin{aligned} Var(V) &= E[(V - EV)^2] \text{ (def.)} \\ &= E\{[cU - E(cU)]^2\} \text{ (subst.)} \\ &= E\{[cU - cEU]^2\} \text{ (property C)} \\ &= E\{c^2[U - EU]^2\} \text{ (algebra)} \\ &= c^2 E\{[U - EU]^2\} \text{ (property C)} \\ &= c^2 Var(U) \text{ (def.)} \end{aligned}$$

Property H

Shifting data over by a constant does not change the amount of variation in them:

$$Var(U + d) = Var(U) \tag{7}$$

for any constant d.

Variance of a Constant: Intuitively, the variance of a constant is 0—after all, it never varies!

Formally:

$$Var(c) = E(c^2) - [E(c)]^2 = c^2 - c^2 = 0 \tag{8}$$

Standard Deviation: The square root of the variance is called the **standard deviation**: $SD(X) = \sqrt{Var(X)}$.

An Alternative Measure of Dispersion

Variance is used historically and mathematically, not because it's the most meaningful measure (of spread).

- Squaring exaggerates larger differences.
- A more natural measure: **mean absolute deviation** (MAD), $E(|U - EU|)$.
- MAD is less mathematically tractable.

The choice of variance allows for powerful mathematical derivations (e.g., Pythagorean Theorem in abstract vector spaces).

Variance of Sum of Independent RVs

If U and V are independent,

$$Var(U + V) = Var(U) + Var(V) \tag{9}$$

Generalizing (9), for constants $a_1, ..., a_k$ and independent random variables $X_1, ..., X_k$, form the new random variable $a_1X_1 + ... + a_kX_k$. Then

$$Var(a_1X_1 + ... + a_kX_k) = \sum_{i=1}^{k} a_i^2 Var(X_i) \tag{10}$$

Importance of Variance

Importance of Variance Properties

- The properties of variance are crucial for understanding the rest of the content.
- Recognize settings where they are applicable. Think of a property like (9) and check for independence.

Central Importance of the Concept of Variance

- The mean is a fundamental descriptor of a random variable.
- Variance is of central importance.
- Used constantly throughout subsequent discussions.
- Next: A quantitative look at variance as a measure of dispersion.

Intuition Regarding the Size of Var(X)

The variance of a random variable X is a measure of dispersion. But, how do we quantify its magnitude?

Chebychev's Inequality: This inequality provides concrete meaning to the concept of variance/standard deviation:

$$P(|X - \mu| \geq c\sigma) \leq \frac{1}{c^2} \tag{11}$$

- For instance, X strays more than 3 standard deviations from its mean at most only 1/9 of the time.
- Used in grading schemes:
  "An A grade is 1.5 standard deviations above the mean" and
  "A C grade is 1.5 standard deviations below the mean"
  (here $c = 1.5$).
- Proof of the inequality provided later.

The Coefficient of Variation

- Reflect on the magnitude of variance.
- E.g., if the price of a widget hovers around a $1 million, but the variation around that figure is only about a dollar, there is essentially no variation. But a variation of about a dollar in the price of a hamburger would be a lot.
- Relate size of $SD(X)$ to $E(X)$ for context.
- Define the coefficient of variation:

$$\text{coef. of var.} = \frac{SD(X)}{EX} = \frac{\sqrt{Var(X)}}{EX}$$

- A scale-free measure to judge the size of variance.

A Useful Fact

For a random variable X:

$$g(c) = E[(X - c)^2] \tag{12}$$

The function $g(c)$ maps a real number c to a real output. What value of c minimizes $g(c)$?

Using the properties of expected value:

$$g(c) = E(X^2) - 2cEX + c^2 \tag{13}$$

Differentiate with respect to c and set to 0 to find:

$$c = EX$$

Optimal Guessing

- Consider guessing people's weights without any prior information.
- Initial inclination: use the mean weight as your guess.
- When you measure the error in your guess using "mean squared error":

  $$E[(X - c)^2]$$

  $c = EX$ minimizes the error.
- This confirms that the optimal guess is the mean weight.

Conclusion and Alternate Consideration

- Plugging $c = EX$ into $g(c)$ shows the minimum value is $E(X - EX)^2$, i.e., $Var(X)$.
- Think of this in terms of long-run average squared error.
- Alternative: minimize average absolute error:

$$E(|X - c|)$$

- The optimal c for this is the median weight.

Indicator Random Variables, and Their Means and Variances

Definition: Indicator Random Variable

Definition
A random variable that has the value 1 or 0, depending on whether a specified event occurs or not, is called an **indicator random variable** for that event.

Handy Facts

- If X is an indicator random variable for event A and $p = P(A)$, then:

$$E(X) = p \tag{14}$$

$$Var(X) = p(1-p) \tag{15}$$

- For example: $EX = P(X = 1) = P(A) = p$.

Application Example

Consider three coins:

- Coin A has $P(heads) = 0.6$
- Coin B (fair) has $P(heads) = 0.5$
- Coin C has $P(heads) = 0.2$

Toss each once (independently), recording heads as X, Y, and Z respectively. $W = X + Y + Z$ is the total number of heads.

Find $P(W = 1)$ and $Var(W)$: To find $P(W = 1)$:

$$\begin{aligned} P(W = 1) &= P(X = 1, Y = 0, Z = 0) + \dots \\ &= 0.6 \times 0.5 \times 0.8 + 0.4 \times 0.5 \times 0.8 + 0.4 \times 0.5 \times 0.2 \\ &= 0.44 \end{aligned}$$

To find $Var(W)$ using indicator random variables:

$$Var(W) = 0.6 \times 0.4 + 0.5 \times 0.5 + 0.2 \times 0.8 = 0.65$$

Example: Return Time for Library Books, Version I

Suppose at a public library:

- Patrons return books exactly 7 days after borrowing.
- Returning to a different branch adds 2 days delay.
- 50% return their books to a "foreign" library.

Find $Var(T)$, where T is the time (either 7 or 9 days) for a book to come back.

Solution:

$$T = 7 + 2I$$

where I is the indicator for the book being returned to a "foreign" branch. $\propto$

$$Var(T) = Var(7 + 2I) = 4Var(I) = 4 \times 0.5 \times (1 - 0.5) = 1.0$$

Example: Return Time for Library Books, Version II

- Borrowers return books after 4, 5, 6 or 7 days with probabilities 0.1, 0.2, 0.3, 0.4.
- 50% return their books to a "foreign" branch (causing 2-day delay).
- Library is open 7 days a week.
- Suppose you wish to borrow a certain book, and inquire at the library near the close of business on Monday.
- You are told that it had been checked out the previous Thursday.
- Assume that no one else is waiting for the book, you check every evening, and a borrower returning to a foreign branch is independent of his/her return day.
- Find the probability of waiting until Wednesday evening.

Example: Return Time for Library Books, Version II

Solution Let B: the time (# of days) needed for the book to arrive back at its home branch,

R: the amount of time it takes for borrowers to return books, and define I as before. Note that $B = R + 2\,I$. Then

$$\begin{aligned} P(B = 6 \mid B > 4) &= \frac{P(B = 6 \text{ and } B > 4)}{P(B > 4)} \\ &= \frac{P(B = 6)}{1 - P(B \leq 4)} = \frac{P(B = 6)}{1 - P(B = 4)} \end{aligned}$$

So, $B = 6$ occurs when "R=6 and I=0" or "R=4 and I=1". Thus,

$$\begin{aligned} P(B = 6 \mid B > 4) &= \frac{P(\text{“}R = 6 \text{ and } I = 0\text{” or “}R = 4 \text{ and } I = 1\text{”})}{1 - P(R = 4 \text{ and } I = 0)} \\ &= \frac{0.3 \cdot 0.5 + 0.1 \cdot 0.5}{1 - 0.1 \cdot 0.5} \\ &= \frac{4}{19} \approx .21 \end{aligned}$$

Simulation Check

```
libsim <- function(nreps) {
   prt <- sample(c(4,5,6,7),nreps,replace=T,
                 prob=c(0.1,0.2,0.3,0.4))
   i <- sample(c(0,1),nreps,replace=T)
   b <- prt + 2*i
   x <- cbind(prt,i,b)
   bgt4 <- x[b > 4,]
   mean(bgt4[,3] == 6)
}
```

Use R's vector operations for easier coding and faster running.

Introduction

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Note: Except for some specific calculations, these apply to both discrete and continuous random variables.

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While the expected value tells us the average value a random variable takes on, we also need a measure of its variability.

In other words, we want a measure of **dispersion**. The classical measure is **variance**.

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In the expression $Var(U) = E[(U - EU)^2]$:

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The term $E(U^2)$ is evaluated using (**??**).

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\begin{aligned}
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&= \dots \\
&= c^2 Var(U) \text{ (definition)}
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Shifting data over by a constant does not change the amount of variation in them:

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- Recognize settings where they are applicable.
- Example: Seeing variance of sum of two random variables? Think of a property like (**??**) and check for independence.

More Practice with the Properties of Variance

Suppose X and Y are independent random variables with given expectations and variances. Let's find $Var(XY)$:

$$\begin{aligned} Var(XY) &= E(X^2Y^2) - [E(XY)]^2 \\ &= E(X^2) \cdot E(Y^2) - (EX \cdot EY)^2 \\ &= [Var(X) + (EX)^2] \cdot [Var(Y) + (EY)^2] - (EX \cdot EY)^2 \\ &= 28 \end{aligned}$$

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Intuition Regarding the Size of Var(X)

The variance of a random variable X is a measure of dispersion. But how do we quantify its significance?

Quote:

> *"A billion here, a billion there, pretty soon, you're talking real money" - Senator Everitt Dirksen*

Chebychev's Inequality

This inequality provides concrete meaning to the concept of variance/standard deviation:

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The Coefficient of Variation

- Reflect on the magnitude of variance.
- Relate size of $Var(X)$ to $E(X)$ for context.
- Define the coefficient of variation:

$$\text{coef. of var.} = \frac{\sqrt{Var(X)}}{EX}$$

- A scale-free measure to judge the size of variance.

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Optimal Guessing

- Consider guessing people's weights without any prior information.
- Initial inclination: guess the mean weight.
- Error measure using mean squared error:

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- Differentiate with respect to c and set to 0 to find:

$$c = EX$$

- This confirms that the optimal guess is the mean weight.

Conclusion and Alternate Consideration

- Plugging $c = EX$ into $g(c)$ shows the minimum value is $E(X - EX)^2$, i.e., $Var(X)$.
- Think of this in terms of long-run average squared error.
- Alternative: minimize average absolute error:

$$E(|X - c|)$$

- The optimal c for this is the median weight.

Covariance

- Covariance measures the degree to which two variables U and V vary together.
- Defined as:

$$Cov(U, V) = E[(U - EU)(V - EV)]$$

- Indicates if two variables are positively or negatively related.

Understanding Covariance

- If both variables are usually large or small together, covariance is positive.
- E.g. Height and weight: Taller people tend to be heavier and vice-versa, indicating a positive covariance.

Properties of Covariance

- Covariance can be rewritten as:

$$Cov(U,V) = E(UV) - EU \cdot EV$$

- Variance relation:

$$Var(U+V) = Var(U) + Var(V) + 2Cov(U,V)$$

- And more generally:

$$Var(aU+bV) = a^2 Var(U) + b^2 Var(V) + 2abCov(U,V)$$

Special Cases and Generalizations

- If $Cov(U, V) = 0$, $Var(U + V) = Var(U) + Var(V)$.
- This relation is analogous to the Pythagorean Theorem in a special mathematical context.
- General formula:

$$Var(a_1X_1+...+a_kX_k) = \sum_{i=1}^{k} a_i^2 Var(X_i)+2 \sum_{1\leq i<j\leq k}^{k} a_i a_j Cov(X_i, X_j)$$

- If X_i are independent:

$$Var(a_1X_1 + ... + a_kX_k) = \sum_{i=1}^{k} a_i^2 Var(X_i)$$

Indicator Random Variables, and Their Means and Variances

Definition: Indicator Random Variable

Definition
A random variable that has the value 1 or 0, depending on whether a specified event occurs or not, is called an **indicator random variable** for that event.

Handy Facts

- If X is an indicator random variable for event A and $p = P(A)$, then:

$$E(X) = p \tag{13}$$

$$Var(X) = p(1 - p) \tag{14}$$

- For example: $EX = P(X = 1) = P(A) = p$.

Application Example

Consider three coins:

- Coin A has $P(\mathit{heads}) = 0.6$
- Coin B (fair) has $P(\mathit{heads}) = 0.5$
- Coin C has $P(\mathit{heads}) = 0.2$

Toss each once, recording heads as X, Y, and Z respectively.
$W = X + Y + Z$ is the total number of heads.

Finding $P(W = 1)$ and $Var(W)$

- To find $P(W = 1)$:

$$
\begin{aligned}
P(W = 1) &= P(X = 1, Y = 0, Z = 0) + \dots \\
&= 0.6 \times 0.5 \times 0.8 + 0.4 \times 0.5 \times 0.8 + 0.4 \times 0.5 \times 0.2
\end{aligned}
$$

- To find $Var(W)$ using indicator random variables:

$$
Var(W) = 0.6 \times 0.4 + 0.5 \times 0.5 + 0.2 \times 0.8
$$

Example: Return Time for Library Books, Version I

Suppose at a public library:

- Patrons return books exactly 7 days after borrowing.
- Returning to a different branch adds 2 days delay.
- 50% return their books to a "foreign" library.

Find $Var(T)$, where T is the time (either 7 or 9 days) for a book to come back.

Solution

$$T = 7 + 2I$$

where I indicates if the book is returned to a “foreign” branch.

$$Var(T) = Var(7 + 2I) = 4Var(I) = 4 \times 0.5 \times (1 - 0.5)$$

Example: Return Time for Library Books, Version II

- Borrowers return books after 4, 5, 6 or 7 days with probabilities 0.1, 0.2, 0.3, 0.4.
- 50% return their books to a "foreign" branch (2-day delay).
- Library open 7 days a week.
- Borrow a book on Monday; checked out previous Thursday. Find the probability of waiting until Wednesday evening.

Solution

Let B be return time, I indicate "foreign" branch.

$$P(B = 6 \mid B > 4) = \frac{0.5 \times 0.3 + 0.5 \times 0.1}{1 - 0.5 \times 0.1}$$
$$= \frac{4}{19}$$

Simulation Check

```
libsim <- function(nreps) {
    prt <- sample(c(4,5,6,7),nreps,replace=T,prob=c(
    i <- sample(c(0,1),nreps,replace=T)
    b <- prt + 2*i
    x <- cbind(prt,i,b)
    bgt4 <- x[b > 4,]
    mean(bgt4[,3] == 6)
}
```

Use R's vector operations for easier coding and faster running.

Introduction

Introduction

- The concepts and properties of variance are fundamental in probability and statistics.
- Applicable to both discrete and continuous random variables.

Definition of Variance

- Variance measures the variability or dispersion of a random variable.
- It's defined as the mean squared difference between a random variable and its mean.
- For a random variable U: $Var(U) = E[(U - E[U])^2]$

Example: Variance of a Die Roll

- Calculate the variance of a die roll, X.
- Variance: $Var(X) = E[(X - 3.5)^2]$
- Interpretation: Measure of dispersion for X.

Variance Calculation

- Calculate $Var(X)$ using the formula.
- Use the expected value properties to simplify.

Properties of Variance

Property F: Variance Formula

- Property F: $Var(U) = E(U^2) - (E[U])^2$
- Formula for variance in terms of the second moment.

Property G: Scaling Variance

- Property G: $Var(cU) = c^2 Var(U)$
- Variance scales with the square of a constant factor.

Property H: Shifting Variance

- Property H: $Var(U + d) = Var(U)$
- Shifting data by a constant doesn't change the variance.

Standard Deviation

- The square root of the variance is the standard deviation.

Conclusion

Conclusion

- Variance is a key measure of dispersion in probability and statistics.
- Understanding variance properties is crucial for future topics.

Introduction

Introduction

- Practice with the properties of variance.
- Independent random variables X and Y.
- Given: $EX = 1$, $EY = 2$, $Var(X) = 3$, and $Var(Y) = 4$.

Variance of XY

- Calculate $Var(XY)$ using properties of variance.
- $Var(XY) = E(X^2 Y^2) - [E(XY)]^2$

Central Importance of Variance

- Variance is a fundamental concept in statistics.
- It measures dispersion and variability.

Intuition Regarding Var(X)

- Understanding the magnitude of variance.
- Relating variance to standard deviation.
- Chebychev's Inequality and coefficient of variation.

Chebychev's Inequality

- $P(|X - \mu| \geq c\sigma) \leq \frac{1}{c^2}$
- Provides bounds on the probability of deviation from the mean.

The Coefficient of Variation

- Coefficient of Variation: $\frac{\sqrt{Var(X)}}{EX}$
- A scale-free measure of variability.

A Useful Fact

Minimizing $E[(X - c)^2]$

- Find the value of c that minimizes $E[(X - c)^2]$.
- Minimizing squared error for random variable X.

Covariance

Covariance

- Introduction to covariance.
- Measure of how two variables vary together.
- $Cov(U, V) = E[(U - EU)(V - EV)]$

Properties of Covariance

- Covariance and correlation.
- The formula for calculating covariance.

Covariance Calculation

- Understanding covariance through an example.
- Interpretation of positive and negative covariance.

The Coefficient of Variation

- Coefficient of Variation: $\frac{\sqrt{Var(X)}}{EX}$
- A scale-free measure for judging variance.

Conclusion

Conclusion

- Variance is a vital concept in statistics.
- Covariance measures the relationship between two variables.
- Coefficient of Variation helps judge the magnitude of variance.

Introduction

Definition
A random variable that takes values 1 or 0, based on the occurrence of a specific event, is known as an **indicator random variable** for that event.

Properties of Indicator Random Variables

Handy facts: *Suppose X is an indicator random variable for the event A. Let p denote $P(A)$.*

$$E(X) = p \tag{1}$$

$$Var(X) = p(1 - p) \tag{2}$$

These properties are essential and can be easily derived.

Example

Example: Tossing Coins

- Coin A: $P(\text{Heads}) = 0.6$
- Coin B: Fair
- Coin C: $P(\text{Heads}) = 0.2$
- Toss A, B, and C once to get X, Y, and Z.
- Let $W = X + Y + Z$ (total heads).

Calculating $P(W = 1)$

- Find $P(W = 1)$ using indicator random variables.

Calculating $Var(W)$

- Calculate $Var(W)$ using properties of indicator random variables.

Conclusion

Conclusion

- Indicator random variables are valuable tools in probability and statistics.
- They help simplify complex calculations.
- Understanding their mean and variance properties is essential.

Introduction

Introduction

- Probability and statistics play a crucial role in understanding randomness and uncertainty.
- This presentation will cover various topics related to distributions, examples, and models.

Distributions

Definition
The **distribution** of a discrete random variable is the set of possible values along with their associated probabilities.

Example: Roll a fair six-sided die.

$$\text{Distribution of } X = \{(1, 1/6), (2, 1/6), (3, 1/6), (4, 1/6), (5, 1/6), (6, 1/6) \quad (3)$$

Probability Mass Function (PMF)

Definition
The **probability mass function (PMF)** of a discrete random variable V, denoted p_V, is defined as:

$$p_V(k) = P(V = k) \tag{4}$$

for any value k in the support of V.

Example: Toss a coin until the first head.

$$p_N(k) = \frac{1}{2^k}, \quad k = 1, 2, \ldots \tag{5}$$

Examples of Distributions

Example: Sum of two fair six-sided dice.

$$p_S(k) = \begin{cases} \frac{1}{36}, & k = 2 \\ \frac{2}{36}, & k = 3 \\ \frac{3}{36}, & k = 4 \\ \dots \\ \frac{1}{36}, & k = 12 \end{cases} \tag{6}$$

Example: Watts-Strogatz Random Graph Model (Degree Distribution).

$$p_M(r) = \frac{\binom{n-3}{r-2}\binom{n^2/2-3n/2-(n-3)}{k-(r-2)}}{\binom{n^2/2-3n/2}{k}} \tag{7}$$

Conclusion

Conclusion

- Probability and statistics provide tools to understand randomness and uncertainty.
- Distributions and probability mass functions help describe the behavior of random variables.
- Examples and models demonstrate how these concepts are applied in various scenarios.