STAT7630: Bayesian Statistics
Lecture Slides # 13

Bayesian Logistic Regression Models
Chapter 13 (Logistic Regression)

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Outline

Bayesian Logistic Regression

- Prediction and Classification
- Evaluating Model Performance
- Bayesian Logistic Regression with Multiple Predictors

Regression for Binary Data

- Consider a regression framework where the response variable Y is binary, taking exactly two values (e.g., Pass/Fail, Survive/Die, Win/Loss), typically encoded as 0 or 1.
- Traditional models, such as the Normal or Poisson regression, are not suitable for this type of response variable due to the binary nature of Y.
- When Y is binary, the expected value $\mathbf{E}(Y)$ corresponds to the probability $P(Y = 1)$.
- The model will establish a relationship between $\mathbf{E}(Y)$ and a predictor X, or a set of predictors $X_1, X_2, \ldots, X_p$, to capture the underlying dependency structure.

Review: Odds and Probability

- Recall that for an event with probability π, the odds of the event are defined as $\frac{\pi}{1-\pi}$.
- Since the probability π ranges from 0 to 1, the odds span values from 0 to ∞.
- The odds are:
  - Less than 1 if and only if $\pi < 0.5$.
  - Equal to 1 if and only if $\pi = 0.5$.
  - Greater than 1 if and only if $\pi > 0.5$.

Real Data Example: Logistic Regression Model

- Consider a dataset of senior citizens where two variables are measured:
  - A binary response variable Y.
  - An (approximately) continuous predictor variable X.
- The response variable Y indicates senility status:
  - $Y = 0$: No senility present.
  - $Y = 1$: Senility present.
- The predictor variable X represents the individual's score on a subset of the Wechsler Adult Intelligence Scale (WAIS) exam.

Real Data Example: Logistic Regression Model

- Recall that for a binary response Y_i, the expected value is $\mathbf{E}(Y_i) = P(Y_i = 1)$.
- We model $\mathbf{E}(Y_i) = \pi_i$ as a function of X_i, the WAIS score for the individual.
- The mean response given the predictors follows:

$$Y_i \mid \beta_0, \beta_1 \overset{\text{ind}}{\sim} \text{Bernoulli}(\pi_i), \quad \text{where } \log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 X_i.$$

- The "linear predictor" $\beta_0 + \beta_1 X_i$ is related to the log-odds that $Y_i = 1$.
- The model equation can also be expressed in terms of the odds or probability:

$$\frac{\pi_i}{1-\pi_i} = e^{\beta_0 + \beta_1 X_i} \quad \text{and} \quad \pi_i = \frac{e^{\beta_0 + \beta_1 X_i}}{1 + e^{\beta_0 + \beta_1 X_i}}.$$

General Form of the Logistic Regression Model

- For a logistic regression model with multiple predictors, the log-odds is modeled as:

$$\log(\text{odds}) = \log\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p.$$

- Interpretation of β_1:
  - Let odds_x be the odds that $Y = 1$ when $X_1 = x$, and let odds_{x+1} be the odds that $Y = 1$ when $X_1 = x + 1$ (a one-unit increase in X_1).
  - Holding all other predictors $X_2, \ldots, X_p$ constant:
    - β_1 represents the expected change in log-odds:

$$\beta_1 = \log(\text{odds}_{x+1}) - \log(\text{odds}_x).$$

    - e^{β_1} represents the expected multiplicative change in odds:

$$e^{\beta_1} = \frac{\text{odds}_{x+1}}{\text{odds}_x}.$$

Priors in the Logistic Regression Model

- To perform Bayesian logistic regression, we must specify priors on the coefficients $\beta_0, \beta_1, \ldots, \beta_p$.
- A common choice is to use normal priors for these coefficients:

$$\beta_j \sim N(\mu_j, \sigma_j^2), \quad j = 0, 1, \ldots, p.$$

- For objective Bayesian analysis, we can set prior means $\mu_j = 0$ for all coefficients.
- Posterior estimation involves sampling techniques, such as:
    - Implementing the Metropolis-Hastings algorithm manually.
    - Using high-level tools like `stan_glm` in the `rstanarm` package.
- **Example:** Applying "noninformative" priors in `R` to the WAIS senility dataset demonstrates this approach.

Specifying Subjective Priors in the Logistic Regression Model

- A structured process for eliciting prior information can be particularly effective when using `stan_glm`.
- In `stan_glm`, the prior is placed on the *centered* intercept (β_0^*), distinct from the β_0 in the model.
- **Example:** Prior elicitation for senility probability.
    - Assume a "typical" subject has a probability of senility between 0.2 and 0.6.
    - Corresponding log-odds range:

$$\log\left(\frac{0.2}{0.8}\right) = -1.4 \quad \text{to} \quad \log\left(\frac{0.6}{0.4}\right) = 0.4.$$

    - Set the prior mean for the **CENTERED** β_0^* to the midpoint:

$$\mu_{\beta_0^*} = \frac{-1.4 + 0.4}{2} = -0.5.$$

    - Set the prior standard deviation to half the range:

$$\sigma_{\beta_0^*} = \frac{0.4 - (-1.4)}{2} = 0.45.$$

More on Specifying Subjective Priors in the Logistic Regression Model

Example: Specifying the prior mean and standard deviation for β_1.

- **Belief:** For a one-unit increase in WAIS score, the odds of senility are expected to fall between 0.5 and 1 (i.e., reduced to half or remain the same).
- Corresponding range for β_1 (log-odds):

$$\beta_1 \in [\log(0.5), \log(1)] = [-0.69, 0]\,.$$

- **Prior elicitation:**
  - Set the prior mean for β_1 to the midpoint of the range:

$$\mu_{\beta_1} = \frac{-0.69 + 0}{2} = -0.35.$$

  - Set the prior standard deviation to half the range:

$$\sigma_{\beta_1} = \frac{0 - (-0.69)}{2} = 0.175.$$

- This prior reflects expert knowledge about the expected effect of WAIS scores on senility odds.

Fitting the Logistic Regression Model

- Priors can be specified, and the `stan_glm` function in the `rstanarm` package automates the Metropolis-Hastings algorithm for posterior sampling.
- Perform standard MCMC diagnostics to ensure model convergence and reliability:
  - Check trace plots, R-hat statistics, and effective sample sizes.
  - Take remedial actions if diagnostics indicate convergence issues.
- Summaries of the posterior distributions for model coefficients can be obtained using:
  - The `summary()` function for detailed statistical summaries.
  - The `tidy()` function for a cleaner, formatted output.
- Refer to `R` examples for implementing and fitting the logistic regression model using Bayesian methods.

Interpretations of Estimated Parameters

- Posterior estimate for β_1:
  - Approximate value: $\widehat{\beta}_1 \approx -0.3$.
  - The exact estimate may vary slightly depending on the choice of priors and the specifics of the MCMC run.
- Interpretation of β_1:
  - The odds of senility decrease by a factor of $e^{-0.3} \approx 0.74$ for each one-point increase in WAIS score.
  - This corresponds to a 26% reduction in the odds of senility per unit increase in WAIS score.
- Credible interval for β_1:
  - 95% credible interval: $(-0.498, -0.142)$.
  - High posterior probability exists that higher WAIS scores are associated with lower odds of senility.

Outline

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Evaluating Model Performance

Bayesian Logistic Regression with Multiple Predictors

Using the Logistic Regression Model for Prediction

- A primary application of the logistic regression model is predicting the binary response Y for new observations.
- Example:
  - For a new senior citizen with a WAIS score of $X = 10$, predict whether the individual is senile.
- Prediction approach:
  - Plug $X = 10$ into the estimated logistic regression model to compute:

    $$\widehat{\mathbf{E}}(Y \mid X = 10),$$

    which is the estimated probability $\widehat{\pi}$ that the person is senile.
  - Decision rule:
    - If $\widehat{\pi} > 0.5$, predict $Y = 1$ (senile).
    - If $\widehat{\pi} \leq 0.5$, predict $Y = 0$ (not senile).
- Note: A cutoff $c \neq 0.5$ can be used to adjust the sensitivity and specificity of the predictions.

Defining a Classification Rule

- Logistic regression can be used to classify an individual into one of two groups: $Y = 0$ or $Y = 1$.
- Classification rule:
  - For a given predictor value x (or a set of predictors $x_1, x_2, \ldots, x_p$), generate a large number of posterior predictions for Y.
  - Let p denote the proportion of posterior predictions where $Y = 1$.
  - Select a classification cutoff value $c \in [0, 1]$.
  - Decision rule:
    - If $p \geq c$, classify the individual into the $Y = 1$ group.
    - If $p < c$, classify the individual into the $Y = 0$ group.
- This approach allows for flexible classification thresholds based on the specific context or desired trade-off between sensitivity and specificity.

Choice of Classification Cutoff Value

- The default classification cutoff value is $c = 0.5$, which is commonly used.
- However, in certain scenarios, a different cutoff value may be more appropriate:
  - Especially when the cost of one type of misclassification error significantly outweighs the cost of the other.
- Example from the book:
  - $Y = 1$: Predicting rain (carry an umbrella).
  - $Y = 0$: Predicting no rain (no umbrella).
  - Decision trade-off:
    - Is it worse to carry an umbrella unnecessarily or to forgo the umbrella and get wet?
  - To minimize the risk of getting wet, we might choose a smaller cutoff, such as $c = 0.25$, thereby predicting rain more often and playing it safe.
- Adjusting c allows for flexibility to match the classification strategy to the specific context and error costs.

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Assessing Model Quality

- The posterior predictive distribution can be used to evaluate model quality.
- Approach:
    - Use the `pp_check` function as a shortcut to generate numerous posterior-simulated datasets.
    - For each simulated dataset, calculate the count of $Y = 1$ values.
    - Visualize these counts using a histogram.
- Model evaluation:
    - Compare the actual count of $Y = 1$ values from the observed data to the distribution of simulated counts.
    - If the observed count falls near the center of the simulated distribution, it indicates that the model fits well.
- **Example:** See `R` implementation for the WAIS dataset.

Measuring Classification Accuracy

- Classification accuracy evaluates the performance of the logistic regression model in correctly classifying binary observations.
- A common approach is to use a **confusion matrix**: Compare the actual binary values (Y) with the predicted binary values ($\widehat{Y}$) based on the chosen classification rule.
- For a sample of n individuals:
  - Let Y_i denote the actual binary outcome for observation i, where $i = 1, \ldots, n$.
  - Compute $\widehat{Y}_i$, the predicted binary outcome, using the fitted logistic regression model and the chosen classification cutoff.
- The confusion matrix summarizes the counts of:
  - True Positives (correctly predicted $Y = 1,\ \widehat{Y} = 1$)
  - True Negatives (correctly predicted $Y = 0,\ \widehat{Y} = 0$).
  - False Positives (incorrectly predicted $Y = 0,\ \widehat{Y} = 1$).
  - False Negatives (incorrectly predicted $Y = 1,\ \widehat{Y} = 0$).

Confusion Matrix

- The confusion matrix summarizes the classification results in a 2×2 format with entries a, b, c, and d:

	$\widehat{Y} = 0$	$\widehat{Y} = 1$
$Y = 0$	a	b
$Y = 1$	c	d

- **Definitions:**
  - a: True Negatives (correctly predicted $Y = 0,\ \widehat{Y} = 0$).
  - b: False Positives (incorrectly predicted $Y = 0,\ \widehat{Y} = 1$).
  - c: False Negatives (incorrectly predicted $Y = 1,\ \widehat{Y} = 0$).
  - d: True Positives (correctly predicted $Y = 1,\ \widehat{Y} = 1$).

Confusion Matrix

- This framework provides metrics such as accuracy, precision, recall, and $F1$-score to assess classification performance.
- Metrics for model performance:
  - **Overall Accuracy**: Proportion of all observations correctly classified:

$$\text{Accuracy} = \frac{a+d}{a+b+c+d}.$$

  - **Misclassification Rate**: Proportion of incorrectly classified observations:

$$\text{Misclassification Rate} = 1 - \text{Accuracy} = \frac{b+c}{a+b+c+d}.$$

Sensitivity and Specificity

- **Sensitivity** (True Positive Rate): Proportion of $Y = 1$ observations correctly classified.

$$\text{Sensitivity} = \frac{d}{c + d}.$$

- **Specificity** (True Negative Rate): Proportion of $Y = 0$ observations correctly classified.

$$\text{Specificity} = \frac{a}{a + b}.$$

- Interpretation:
  - **Sensitivity** measures how well the model identifies true positives.
  - **Specificity** measures how well the model identifies true negatives.
- These metrics are crucial for evaluating model performance, especially when the costs of false positives and false negatives differ.

Aims of Sensitivity and Specificity

- Ideally, both **sensitivity** and **specificity** should be high to ensure robust model performance.
- However, practical applications often dictate prioritizing one over the other.
- **Example:** Medical testing for a potentially deadly disease (e.g., breast cancer).
  - **High Sensitivity**:
    - Ensures that most true cases of the disease ($Y = 1$) are detected.
    - Reduces the risk of a true cancer going undiagnosed, preventing untreated conditions.
  - **Lower Specificity**:
    - May result in some healthy individuals ($Y = 0$) being misclassified as sick.
    - This could lead to wasted time and resources but does not carry deadly consequences.
- In this context, high sensitivity is often more critical to minimize life-threatening errors.

Tuning the Classification Rule Based on Sensitivity and Specificity

- The classification rule is determined by the cutoff value c.
- To optimize the model's performance:
  - Experiment with various values of c.
  - For each c, compute the **in-sample** sensitivity and specificity using the resulting confusion matrix.
  - Alternatively, use **cross-validation** to estimate sensitivity and specificity for **out-of-sample** predictions.
- Trade-off between sensitivity and specificity:
  - **Lower** c: Increases sensitivity but decreases specificity.
  - **Higher** c: Increases specificity but decreases sensitivity.
- This trade-off should be considered in the context of the application to balance the costs of false positives and false negatives.

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Bayesian Logistic Regression with Multiple Predictors

- The logistic regression model can be extended to include multiple predictors $X_1, X_2, \ldots, X_p$.
- **Example:** Predicting whether it rains tomorrow in Perth, Australia (Y is binary).
  - **Predictors:**
    - X_1: Humidity at 9 a.m. today.
    - X_2: Humidity at 3 p.m. today.
    - X_3: Whether it rains today (binary).
- Model equation for the mean response:

$$\pi_i = \mathbf{E}(Y_i \mid \mathbf{x}) = \frac{\exp(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3})}{1 + \exp(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3})}.$$

- **Bayesian framework:**
  - Priors are specified for $\beta_0, \beta_1, \beta_2, \beta_3$.
  - Posterior inference is conducted to estimate parameters and make predictions.

Fitting Bayesian Multiple Logistic Regression

- **Priors:** Normal priors for the coefficients $\beta_0, \beta_1, \beta_2, \beta_3$ are specified as usual.
- Posterior simulation conducted via:
  - Direct Metropolis-Hastings.
  - Automated tools like `stan_glm` from the `rstanarm` package.
- Example of estimated coefficients:
  $\widehat{\beta}_1 = -0.007$, $\widehat{\beta}_2 = 0.08$, $\widehat{\beta}_3 = 1.15$. (Estimates may vary depending on prior specifications and MCMC runs.)
- **Inference:**
  - The 95% credible interval for β_1 includes 0 suggesting that "humidity at 9 a.m. today" may not be necessary as a predictor.
  - Strong association between predictors (X_1, X_2, X_3) might explain why not all predictors are required.
- **Interpretation:** Model refinement can be based on credible intervals and collinearity considerations.

Model Selection in Bayesian Multiple Logistic Regression

- Model selection involves comparing different sets of predictors using standard criteria:
  - **Cross-Validation (CV) Accuracy**: Measures prediction performance on held-out data.
  - **Expected Log Predictive Density (ELPD)**: Reflects the model's fit to future data.
  - **Bayesian Information Criterion (BIC)**: Balances model fit and complexity.

Model Selection in Bayesian Multiple Logistic Regression

- **Example:** Predicting rain with multiple predictors.
  - Compare:
    - Model with 3 predictors (X_1, X_2, X_3).
    - Model with only X_1.
  - **Results:**
    - The 3-predictor model shows better CV accuracy, higher ELPD, and lower BIC.
    - Therefore, the 3-predictor model is preferred over the single-predictor model.
  - Refinement:
    - A model with only X_2 and X_3 (excluding X_1) slightly outperforms the 3-predictor model based on these criteria.
- **Conclusion:**
  - Model selection criteria help identify a balance between complexity and predictive performance.

```
    confusion <- matrix(0, nrow = 2, ncol = 2, dimnames = list(c("0", "1"), c("0", "1")))
  }

  # Compute accuracy
  accuracy <- sum(diag(confusion)) / sum(confusion)
  return(accuracy)
}
```

```
library(rstanarm)
library(tidyverse)

# Perform k-fold cross-validation
CV_acc_vals <- sapply(
  folds,
  function(test_ind) {
    train_ind <- setdiff(seq_len(nrow(wais_data)), test_ind)
    comp_CV_acc(train_ind, test_ind, wais_mod, wais_data, cutoff = 0.5)
  }
)
```

```
# Calculate mean cross-validated accuracy
CV_acc <- mean(CV_acc_vals)
cat("Cross-Validated Classification Accuracy:", CV_acc, "\n")
```

```
## Cross-Validated Classification Accuracy: 0.8266667
```

Bayesian Logistic Regression with Multiple Predictors

```
# Rain example from the book

# Load required libraries
library(rstanarm)      # For Bayesian logistic regression
library(tidyverse)     # For data manipulation and visualization
library(broom.mixed)   # For tidy summaries of Bayesian models
library(bayesrules)    # For Bayesian tools
library(caret)         # For stratified cross-validation

# Load and process the data
data(weather_perth)  # Assuming `weather_perth` is preloaded
weather <- weather_perth %>%
  select(day_of_year, raintomorrow, humidity9am, humidity3pm, raintoday)
```

Prior belief for logistic regression: On a "typical" day, the chance of rain is 20% (0.2). The prior mean on the CENTERED beta_0 (intercept) is log(0.2/(1 - 0.2)) = -1.4. A prior SD of 0.7 implies a 95% chance the log-odds are between -2.8 and 0. This corresponds to odds of 0.06 to 1, or probabilities between 0.057 and 0.5.

```
# Fit a Bayesian logistic regression model with multiple predictors
rain_stanglm2 <- stan_glm(
```

```
  raintomorrow ~ humidity9am + humidity3pm + raintoday,
  data = weather,
  family = binomial,                                # Logistic regression
  prior_intercept = normal(-1.4, 0.7),             # Prior for intercept
  prior = normal(0, 2.5, autoscale = TRUE),        # Weakly informative prior for coefficients
  chains = 4,                                      # Number of MCMC chains
  iter = 10000                                     # Number of iterations (post-warmup = 5000)
)

# Summarize posterior estimates with confidence intervals
rain_stanglm2_summ <- tidy(rain_stanglm2, effects = "fixed", conf.int = TRUE, conf.level = 0.95)
print(rain_stanglm2_summ)

# Model comparison: Fit a simpler model with a single predictor
rain_stanglm1 <- stan_glm(
  raintomorrow ~ humidity9am,
  data = weather,
  family = binomial,
  prior_intercept = normal(-1.4, 0.7),             # Same prior for intercept
  prior = normal(0.07, 0.035),                    # Prior for slope (adjusted based on context)
  chains = 4,
  iter = 10000,
  prior_PD = FALSE                                # Use posterior data
)
```

```
# Compare classification accuracy using k-fold cross-validation
# The book suggests c = 0.2 as a reasonable cutoff, but feel free to explore others
set.seed(123)  # For reproducibility

# Cross-validation for rain_stanglm1
CV_acc_1 <- classification_summary_cv(
  model = rain_stanglm1,
  data = weather,
  cutoff = 0.2,
  k = 10  # 10-fold cross-validation
)

# Cross-validation for rain_stanglm2
CV_acc_2 <- classification_summary_cv(
  model = rain_stanglm2,
  data = weather,
  cutoff = 0.2,
  k = 10
)

# Print cross-validated classification accuracy for both models
cat("Cross-Validated Accuracy for Model 1 (Single Predictor):\n")
```

```
## Cross-Validated Accuracy for Model 1 (Single Predictor):
```

```
CV_acc_1$cv
```

```
##   sensitivity specificity overall_accuracy
```

```
## 1   0.6353766   0.7156357            0.701
```

```r
cat("Cross-Validated Accuracy for Model 2 (Multiple Predictors):\n")
```

```
## Cross-Validated Accuracy for Model 2 (Multiple Predictors):
```

```r
CV_acc_2$cv
```

```
##   sensitivity specificity overall_accuracy
## 1   0.7555544   0.8143257            0.802
```

```r
# One approach to model selection:
# LOO for rain_stanglm1 (Single Predictor)
loo1 <- loo(rain_stanglm1)
cat("LOO Estimates for Model 1 (Single Predictor):\n")
```

```
## LOO Estimates for Model 1 (Single Predictor):
```

```r
print(loo1$estimates)
```

```
##             Estimate         SE
## elpd_loo -437.076458 19.0039708
## p_loo       2.217106  0.1798331
## looic     874.152915 38.0079416
```

```r
# LOO for rain_stanglm2 (Multiple Predictors)
loo2 <- loo(rain_stanglm2)
cat("LOO Estimates for Model 2 (Multiple Predictors):\n")
```

```
## LOO Estimates for Model 2 (Multiple Predictors):
```

```r
print(loo2$estimates)
```

```
##            Estimate         SE
## elpd_loo -356.91119 20.8083907
## p_loo       4.23658  0.3494051
## looic     713.82239 41.6167814
```

```r
# Comparing LOO-CV:
cat("\nModel Comparison using LOO:\n")
```

```
##
## Model Comparison using LOO:
```

```r
loo_comp <- loo_compare(loo1, loo2)
print(loo_comp)
```

```
##               elpd_diff se_diff
## rain_stanglm2   0.0       0.0
## rain_stanglm1 -80.2      13.5
```

```r
# Frequentist approach: Use Bayesian Information Criterion (BIC)
# BIC does not incorporate prior information, allowing direct comparison of models

# Fit frequentist logistic regression models
rain_glm1 <- glm(
  raintomorrow ~ humidity9am,
  data = weather,
  family = binomial(logit)
)

rain_glm2 <- glm(
  raintomorrow ~ humidity9am + humidity3pm + raintoday,
  data = weather,
  family = binomial(logit)
)

# Calculate BIC for both models
BIC1 <- BIC(rain_glm1)
BIC2 <- BIC(rain_glm2)

cat("\nBIC Comparison:\n")
```

```
##
## BIC Comparison:
```

```r
cat("Model 1 (Single Predictor): BIC =", BIC1, "\n")
```

```
## Model 1 (Single Predictor): BIC = 883.7219
```

```r
cat("Model 2 (Multiple Predictors): BIC =", BIC2, "\n")
```

```
## Model 2 (Multiple Predictors): BIC = 733.167
```

```r
# Interpretation:
# Lower BIC indicates better model fit while penalizing for model complexity.
# Compare BIC values to decide which model is more appropriate.
```

Bayesian Logistic Regression Model with Simplified Predictors — only humidity3pm & raintoday as Predictors

```r
rain_stanglm_simp <- stan_glm(
  raintomorrow ~ humidity3pm + raintoday,
  data = weather,
  family = binomial,                              # Logistic regression
  prior_intercept = normal(-1.4, 0.7),           # Prior for intercept
  prior = normal(0, 2.5, autoscale = TRUE),      # Weakly informative priors
  chains = 1,                                    # Single MCMC chain for simplicity
  iter = 10000                                   # Total iterations (post-warmup = 5000)
)
```

```r
# Cross-validated classification accuracy (cutoff = 0.2, 10-fold CV)
CV_acc_simp <- classification_summary_cv(
  model = rain_stanglm_simp,
  data = weather,
  cutoff = 0.2,
  k = 10
)

# Print cross-validated accuracy
cat("Cross-Validated Classification Accuracy (Simplified Model):\n")
```

```
## Cross-Validated Classification Accuracy (Simplified Model):
```

```r
print(CV_acc_simp$cv)
```

```
##   sensitivity specificity overall_accuracy
## 1    0.759782   0.8097739              0.8
```

```r
# Evaluate model using Leave-One-Out Cross-Validation (LOO)
loo_simp <- loo(rain_stanglm_simp)
cat("\nLOO Estimates for Simplified Model:\n")
```

```
##
## LOO Estimates for Simplified Model:
```

```r
print(loo_simp$estimates)
```

```
##             Estimate         SE
## elpd_loo -356.229479 20.7998838
## p_loo       3.122271  0.2752114
## looic     712.458958 41.5997676
```

```r
# Frequentist logistic regression model with the same predictors
rain_glm_simp <- glm(
  raintomorrow ~ humidity3pm + raintoday,
  data = weather,
  family = binomial(logit)
)

# Calculate BIC for the simplified model
BIC_simp <- BIC(rain_glm_simp)
cat("\nBIC for Simplified Model (Frequentist):\n", BIC_simp, "\n")
```

```
##
## BIC for Simplified Model (Frequentist):
##  727.1193
```

```r
# Comparison Notes:
# - LOO: Lower `elpd_loo` values indicate a better Bayesian model fit.
# - BIC: Lower BIC indicates a better frequentist model fit while penalizing complexity.
```

Multiple Logistic Regression Model using Base R

```r
# Load required package
library(mvtnorm)  # For multivariate normal distributions

# Extract variables from the weather dataset
rain_tom <- weather$raintomorrow
humid9am <- weather$humidity9am
humid3pm <- weather$humidity3pm
rain_today <- weather$raintoday

# Convert variables to numeric for calculations
rain_tom <- as.numeric(rain_tom) - 1  # Convert rain_tom to 0's and 1's
rain_today <- as.numeric(rain_today)    # Ensure rain_today is numeric

# Construct the design matrix (X) with an intercept
X <- cbind(rep(1, times = length(humid9am)), humid9am, humid3pm, rain_today)

# Prior specifications
beta_pri_mean <- c(0, -0.35, -0.2, 0.5)  # Prior means for beta parameters
# Prior reflects:
# - No specific belief about beta_0 (intercept)
# - Moderate beliefs about the slopes for humid9am, humid3pm, and rain_today
beta_pri_cov <- diag(c(100, 40, 40, 40))  # Prior covariance matrix

# Scaling factor for proposal covariance matrix (can be tuned for MCMC acceptance rates)
k <- 1

# Frequentist Logistic Regression Model (using glm)
log_reg_out <- glm(rain_tom ~ humid9am + humid3pm + rain_today, family = binomial(logit))

# Summary of the logistic regression model
cat("\nSummary of the Logistic Regression Model:\n")
```

```
##
## Summary of the Logistic Regression Model:
```

```r
summary(log_reg_out)
```

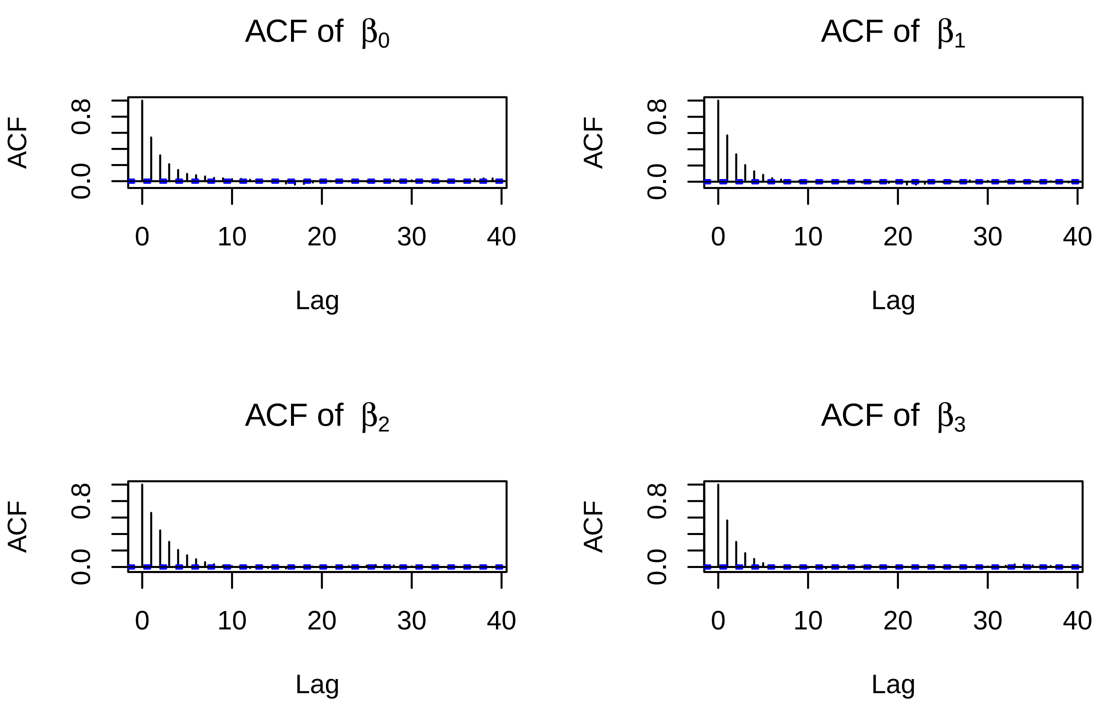
```
##
## Call:
## glm(formula = rain_tom ~ humid9am + humid3pm + rain_today, family = binomial(logit))
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -6.639653   0.488528 -13.591  < 2e-16 ***
## humid9am    -0.006850   0.007407  -0.925    0.355
## humid3pm     0.079831   0.008576   9.309  < 2e-16 ***
## rain_today   1.155394   0.216950   5.326 1.01e-07 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 960.74  on 999  degrees of freedom
## Residual deviance: 705.54  on 996  degrees of freedom
## AIC: 713.54
##
## Number of Fisher Scoring iterations: 5
```

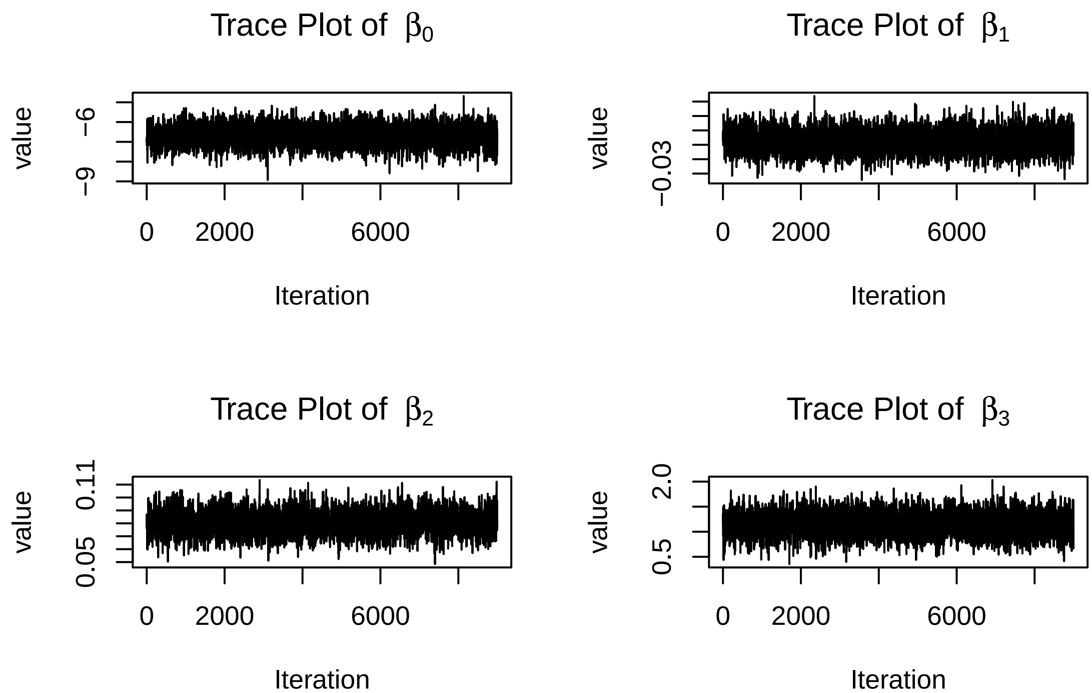
```r
# Set up the proposal covariance matrix
pro_cov_mat <- k * solve(t(X) %*% diag(fitted(log_reg_out)) %*% X)
# Alternative option for proposal covariance matrix:
# pro_cov_mat <- k * diag(ncol(X))

# Initialize MCMC parameters
betas_curr <- beta_pri_mean                # Initial parameter estimates
V <- pro_cov_mat                     # Proposal covariance matrix
mu <- beta_pri_mean                        # Prior mean vector
Sig_inv <- solve(beta_pri_cov)           # Inverse of prior covariance matrix

# MCMC settings
j <- 0                                      # Counter for accepted proposals
burn <- 1000                               # Number of burn-in iterations
Niter <- 50000                             # Total number of iterations
mcmc_res <- matrix(0, nrow = Niter, ncol = length(beta_pri_mean))  # Storage for MCMC samples

# MCMC sampling loop
for (i in 1:Niter) {
  # Generate candidate beta from proposal distribution
  betas_pro <- rmvnorm(1, betas_curr, V)
  betas_pro <- as.vector(betas_pro)

  # Calculate the Metropolis ratio
  log_ratio <- (
    -k * sum(log(1 + exp(X %*% betas_pro))) + sum(rain_tom * X %*% betas_pro) -
      0.5 * t(betas_pro - mu) %*% Sig_inv %*% (betas_pro - mu)
  ) - (
    -k * sum(log(1 + exp(X %*% betas_curr))) + sum(rain_tom * X %*% betas_curr) -
      0.5 * t(betas_curr - mu) %*% Sig_inv %*% (betas_curr - mu)
  )

  # Accept/reject step
  if (runif(1) < exp(log_ratio)) {
    betas_curr <- betas_pro  # Accept candidate
    j <- j + 1               # Increment acceptance counter
  }

  # Save the current beta values
  mcmc_res[i, ] <- betas_curr
}

# Calculate acceptance rate
acc_rate <- j / Niter
cat("Acceptance Rate:", acc_rate, "\n")
```

```
## Acceptance Rate: 0.49968
```

```r
# Thinning (everrain_tom 5th value)
thin <- 5
beta_vals_thin <- mcmc_res[seq(1, Niter, by = thin), ]

# Remove burn-in samples
beta_vals_thin_b <- beta_vals_thin[-(1:burn), ]

# Diagnostic plots: Autocorrelation for each parameter
par(mfrow = c(2, 2))  # Set up a 2humid3pm plotting grid
for (i in 1:ncol(beta_vals_thin_b)) {
  acf(beta_vals_thin_b[, i], main = bquote("ACF of " ~ beta[.(i - 1)]) )
}
```




```r
par(mfrow = c(1, 1))  # Reset plotting grid

# Diagnostic plots: Trace plots for each parameter
par(mfrow = c(2, 2))  # Set up a 2humid3pm plotting grid
for (i in 1:ncol(beta_vals_thin_b)) {
  plot(beta_vals_thin_b[, i], type = 'l',
       main = bquote("Trace Plot of " ~ beta[.(i - 1)]),
       xlab = "Iteration", ylab = "value")
}
```




```r
par(mfrow = c(1, 1))  # Reset plotting grid

# Posterior summaries
post_meds <- apply(beta_vals_thin_b, 2, median)      # Posterior medians
post_low <- apply(beta_vals_thin_b, 2, quantile, probs = 0.025)  # 2.5% quantile
post_up <- apply(beta_vals_thin_b, 2, quantile, probs = 0.975)  # 97.5% quantile

# Combine posterior summaries into a tidrain_tom data frame
names_preds <- c("humid9am", "humid3pm", "rain_today")  # Predictor names
beta_post_summ <- data.frame(
  `0.025 Quantile` = post_low,
  `0.5 Quantile` = post_meds,
  `0.975 Quantile` = post_up,
  row.names = c("Intercept", names_preds)
)

# Print the posterior Summary
cat("Posterior Summary:\n")
```

```
## Posterior Summary:
```

```r
print(beta_post_summ)
```

```
##              X0.025.Quantile X0.5.Quantile X0.975.Quantile
## Intercept        -7.66201004   -6.66399335    -5.758651522
## humid9am         -0.02158058   -0.00704039     0.007749144
## humid3pm          0.06356771    0.08053736     0.098286821
## rain_today        0.72117530    1.15555140     1.583379383
```