

Lecture 9 - Stochastic Processes & Markov Chains

①

Stochastic Process (SP): A SP in discrete time

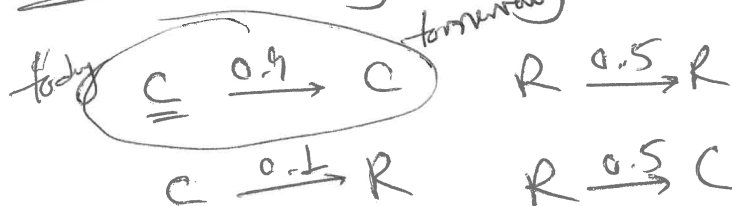
$t \in \{0, 1, 2, \dots\}$ is a sequence of time-indexed RV's $X_0, X_1, \dots$ with $\mathcal{X} = \{X_t, t \geq 0\}$
 RV at time $t=0$ RV at time $t=1$ etc.

- If t is continuous then cts SP
- " " " discrete " discrete "

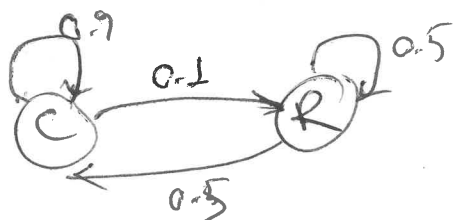
Markov Chains (MC) $\rightarrow$ Discrete Time MC (DTMC)
 $\rightarrow$ Cts $\neq$ MC (CTMC)

Markovian Property: The transition to the next state depends only on the current state.

Ex: Tracking Daily weather $\mathcal{X} = \{ \underbrace{\text{Clear}}_C, \underbrace{\text{Rainy}}_R \}$



Assumption: a day's weather depends only on the previous day



Markov Chain

Distribution of escaping a state is Geo(p).?

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A Stochastic Process with Markovian property is called a Markov Chain.

Discrete Time Markov Chain (DTMC): A SP

$X = \{X_t, t \geq 0, t \text{ discrete}\}$ is a DTMC if for all t

$$P(X_{t+1}=j | X_t=i, X_{t-1}=a, X_{t-2}=b, \dots) =$$

$$P(X_{t+1}=j | X_t=i) = P_{ij} \leftarrow \text{one-step transition probability (from } i \text{ to } j)$$

Markovian Property: (recall) A transition to state $t+1, X_{t+1}$, only depends on the current state X_t .

$$P_{ij} = P(i \rightarrow j)$$

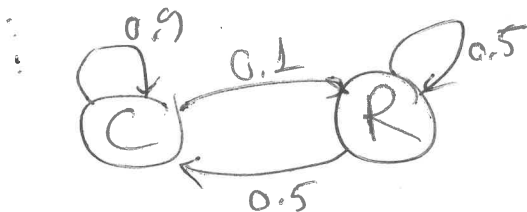
eg. today tomorrow

Ex: Recall daily weather $X = \{C, R\}$

X_1 : weather on day 1 $\in X$

X_2 : " " " 2 $\in X$

X_3 : " " " 3 $\in X$



eg.
 $C \xrightarrow{0.9} C$
 clear today clear tomorrow

$$P_{CC} = 0.9$$

$$P_{RC} = 0.5$$

$$P_{CR} = 0.1$$

$$P_{RR} = 0.5$$

$$\leftarrow P_{ij} = P(i \rightarrow j)$$

$$P = \begin{matrix} & \begin{matrix} C & R \end{matrix} \\ \begin{matrix} C \\ R \end{matrix} & \begin{bmatrix} P_{CC} & P_{CR} \\ P_{RC} & P_{RR} \end{bmatrix} \end{matrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \leftarrow \text{transition matrix} \quad (3)$$

rows of P sum to 1:

$$P_{CC} + P_{CR} = 0.9 + 0.1 = 1$$

$$P_{RC} + P_{RR} = 0.5 + 0.5 = 1$$

$$P_{ij} = P(\text{row } i \rightarrow \text{col } j) \\ = P(i \rightarrow j)$$

$$\text{In general } \sum_j P_{ij} = 1 \Rightarrow \sum_j P(i \rightarrow j) = P(i \rightarrow \text{any})$$

(P_{ij}^2) : Prob ($i \rightarrow j$ in 2 steps)

$$(P_{ij}^2) = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \times \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} (P_{00}^2) & (P_{01}^2) \\ (P_{10}^2) & (P_{11}^2) \end{bmatrix} = P^2$$

$$\text{e.g. } (P_{00}^2) = \underline{P_{00}} \cdot \underline{P_{00}} + \underline{P_{01}} \underline{P_{10}}$$

In general (P_{ij}^n) = Prob ($i \rightarrow j$ in n steps)
 (path does not matter)
 $= P^n \leftarrow$ multiply P with itself n times.

If there are $k+1$ states; $0, 1, \dots, k$

$$P^n = \begin{bmatrix} (P^n)_{00} & (P^n)_{01} & \dots & (P^n)_{0k} \\ \vdots & \vdots & \ddots & \vdots \\ (P^n)_{k0} & (P^n)_{k1} & \dots & (P^n)_{kk} \end{bmatrix}$$

Ex: $P^2 = \begin{bmatrix} 0.86 & 0.14 \\ 0.70 & 0.30 \end{bmatrix}$

$P^5 = \begin{bmatrix} 0.83504 & 0.16496 \\ 0.82480 & 0.17520 \end{bmatrix}$

$P^{10} = \begin{bmatrix} 0.8333508 & 0.1666492 \\ 0.8332460 & 0.1667540 \end{bmatrix}$

$\dots P^{100} = \begin{bmatrix} 0.8\bar{3} & 0.1\bar{6} \\ 0.8\bar{3} & 0.1\bar{6} \end{bmatrix}$

Ex: $X = \{C, S, R\}$
 ↑ ↑ ↑
 clear sunny rainy

$C \xrightarrow{0.7} C$
 $C \xrightarrow{0.2} S$
 $C \xrightarrow{0.1} R$

$S \xrightarrow{0.6} C$
 $S \xrightarrow{0.3} S$
 $S \xrightarrow{0.1} R$

$R \xrightarrow{0.4} C$
 $R \xrightarrow{0.4} S$
 $R \xrightarrow{0.2} R$

$P = \begin{matrix} & C & S & R \\ \begin{matrix} C \\ S \\ R \end{matrix} & \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.6 & 0.3 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{bmatrix} \end{matrix}$

$\rightarrow P^5 = \begin{bmatrix} 0.64195 & 0.24690 & 0.11111 \\ 0.64198 & 0.24691 & 0.11111 \\ 0.64198 & 0.24700 & 0.11102 \end{bmatrix}$

$\rightarrow P^{10} = \begin{bmatrix} 0.6419753 & 0.2469136 & 0.11111 \\ \text{⚡} & \text{⚡} & \text{⚡} \\ \text{⚡} & \text{⚡} & \text{⚡} \end{bmatrix}$

In general, for large n ,

$P^n = \begin{bmatrix} \pi_1 & \pi_2 & \dots & \pi_k \\ \pi_1 & \pi_2 & \dots & \pi_k \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \dots & \pi_k \end{bmatrix}$

Sum = 1

Sum = 1

Sum = 1

Each row is the same

i.e. Prob of ending up at state j after sufficiently large # of steps does not depend on the initial state or first few states

$\lim_{n \rightarrow \infty} (P^n)_{ij} = \pi_j \leftarrow \text{does not depend on } i$

⑤

$\sum_j \pi_j = 1$, so $\vec{\pi} = (\pi_1, \dots, \pi_k)$ is a pmf.

$$\pi_j = \lim_{n \rightarrow \infty} (P_{ij}^n) = \lim_{n \rightarrow \infty} P(i \rightarrow j \text{ in } n \text{ steps})$$

= long term prob. of being in state j
 = limiting prob of being in state j

$\vec{\pi}$ is called limiting distr. or stationary distr.

Goal: How to find $\vec{\pi}$?

Application: mean or average state
 $= \sum_{i=1}^k i \pi_i$ (in the long run)

Recall that for large n

$$P^{n+1} = P^n = \begin{pmatrix} \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \\ \vdots \longrightarrow \\ \vec{\pi} \longrightarrow \end{pmatrix} \quad \leftarrow \text{each row is vector } \vec{\pi}$$

$$\Rightarrow P^n P = \begin{pmatrix} \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \\ \vdots \longrightarrow \\ \vec{\pi} \longrightarrow \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \\ \vdots \longrightarrow \\ \vec{\pi} \longrightarrow \end{pmatrix} P = \begin{pmatrix} \vec{\pi} \longrightarrow \\ \vec{\pi} \longrightarrow \\ \vdots \longrightarrow \\ \vec{\pi} \longrightarrow \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{\pi} P = \vec{\pi}}$$

Ex: $X = \{C, R\}$

$$\vec{\pi} = (\pi_C, \pi_R)$$

$$(\pi_C \ \pi_R) \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (\pi_C \ \pi_R)$$

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both eqns yield this

$$\Rightarrow \left. \begin{aligned} 0.9\pi_C + 0.5\pi_R &= \pi_C \\ 0.1\pi_C + 0.5\pi_R &= \pi_R \end{aligned} \right\} \Rightarrow \begin{aligned} 0.5\pi_R &= 0.1\pi_C \\ 5\pi_R &= \pi_C \end{aligned}$$

$$\Delta \pi_C + \pi_R = 1 \quad (\text{since } \vec{\pi} \text{ is a pmf})$$

$$\Rightarrow 5\pi_R + \pi_R = 1 \Rightarrow \pi_R = 1/6$$

$$\Rightarrow \pi_C = 5(1/6) = 5/6$$

$$\Rightarrow \underline{\underline{\vec{\pi} = (5/6 \quad 1/6)}}$$

Recap:

$$\vec{\pi} P = \vec{\pi}$$

The distribution $\vec{\pi}$ that solves this equation is called the stationary distr. of that M.C.

Thm: If limiting distr. exists, it is also the unique stationary distr.

Ex: $\pi_R = 1/6$ $P(\text{rains on day } 400) = 1/6$
 $P(\text{ " " " } 401) = 1/6$

but $P(R \text{ on } 401 | R \text{ on } 400) = P_{RR} = 0.5$

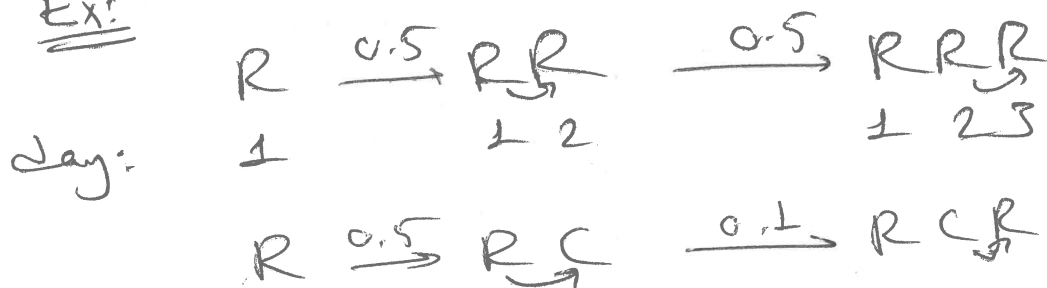
check: $P(R \text{ on } 401) = P(R \text{ on } 401 | R \text{ on } 400)P(R \text{ on } 400) + P(R \text{ on } 401 | C \text{ on } 400)P(C \text{ on } 400)$

$$= 0.5(1/6) + 0.1(5/6) = 2 \times \frac{0.5}{6}$$

$$= \underline{\underline{1/6}}$$

Ex: Find $\vec{\pi}$ for $X = \{C, S, R\}$ example.
 (left as exercise)

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Ex:

prob only depends on the previous state (ordering)

Ex: Umbrella Problem: Prof X commutes between Home & office.

Suppose Prof X has 2 umbrellas.

If it rains, he takes one with him (if available)
 prob (rain) = p & prob (no rain) = $1-p$

Let $i=0,1,2$ be the # of umbrellas he has
 @ home or @ office in a given day.

Qn: (i) Prob of he gets wet? (ii) long term prob of ending up with zero umbrellas

of Umbrellas @ home

$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{pmatrix} \end{matrix}$

$\vec{\pi} = (\pi_0, \pi_1, \pi_2)$

$\vec{\pi} P = \vec{\pi}$

$\Rightarrow (\pi_0 \ \pi_1 \ \pi_2) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{pmatrix} = (\pi_0 \ \pi_1 \ \pi_2)$

$\Rightarrow \pi_2(1-p) = \pi_0$

$(1-p)\pi_1 + p\pi_2 = \pi_1 \Rightarrow \pi_1 = \pi_2$

$\pi_0 + p\pi_1 = \pi_2$

$\Rightarrow \pi_2(1-p) + p\pi_2 = \pi_2$
 $\Rightarrow \pi_2 = \pi_1 = \frac{1}{3-p}$

(ii) $\pi_0 = \frac{1-p}{3-p}$

(i) $P(\text{wet}) = \pi_0 \cdot p = \frac{(1-p)p}{3-p}$

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Random Walk on integers starting at 0

... (-2) (-1) (0) (1) (2) ...

$$\text{Prob}(\text{left}) = \text{Prob}(\text{right}) = \frac{1}{2}$$

