STAT 7650: Homework 1
(Due: Thursday, 02/06/2025)

Note: Show all your work for the necessary steps to receive full credit.

Please turn in the HW on paper, hand-written and/or typed. For computational problems, return only the relevant parts of the output with comments/annotations. From the code you are using to answer the problems, turn in the relevant output, and the figures (if requested), preferably printed from the output. No need to turn in your code or long lists of generated samples.

Please disclose any us of AI in your solutions. Regardless of you use it or not, make sure you submit your own work, not copy from other source(s). Any suspicion of AI use will result in automatic 0 or substantial point loss in any question.

Q1. Let $X_1, X_2, \dots, X_n$ be a random sample from a Bernoulli distribution (i.e. $X_i \overset{iid}{\sim} \text{Ber}(p)$, for $i = 1, \dots, n$, where "Ber" denotes a Bernoulli distribution) with pmf:

$$P(X = x) = p^x(1-p)^{1-x}, \quad x \in \{0, 1\}, \quad 0 \leq p \leq 1.$$

(a) Derive the **likelihood function** $L(p)$ and the **log-likelihood function** $\ell(p)$.

(b) Suppose a random sample of $n = 100$ is observed from the Ber(p) distribution and let $y = \sum_{i=1}^{n} x_i$. Plot the **likelihood function** and **log-likelihood function** for $p \in (0, 1)$ for a sample of size $n = 100$ with Bernoulli observations when: (i) $y = 25$; (ii) $y = 50$; and (iii) $y = 75$

(c) In each plot in part (b), visually estimate the value of p that maximizes $L(p)$.

(d) For each case in part (b), compute the integral of the likelihood function $L(p)$ over $p \in (0, 1)$ using the `integrate` function in R.

Q2. Let $X_1, X_2, \dots, X_n$ be a random sample from a Bernoulli distribution, Ber(p), as in Q1.

(a) Derive the **maximum likelihood estimator (MLE)** of p using calculus.

(b) Derive the **Fisher Information** $I(p)$ for n Bernoulli observations.

(c) For each case in Q1(b), compute the MLE estimate for p based on the given sample.

(d) For each case in Q1(b), compute the **Fisher Information** $I(p)$ for the given sample.

Q3. Suppose $X_1, X_2, \dots, X_n$ are independent Bernoulli random variables, where each X_i follows a Bernoulli distribution with its own parameter p_i: (i.e. $X_i \overset{ind}{\sim} \text{Ber}(p_i)$, for $i = 1, \dots, n$).

(a) Derive the **likelihood function** $L(\mathbf{p}) = L(p_1, p_2, \dots, p_n)$ and the **log-likelihood function** $\ell(\mathbf{p})$.

(b) Derive the **Fisher Information Matrix** for the vector parameter $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

Q4. (Probit Regression) Suppose we have $Y_1, \ldots, Y_n$ are independent (not iid) binary observations. Specifically, $Y_i \sim \text{Ber}(\Phi(x_i^\top \boldsymbol{\theta}))$, for $i = 1, \ldots, n$, $x_1, \ldots, x_n$ are fixed d-dimensional covariates, $\boldsymbol{\theta}$ is a d-dimensional parameter vector, and Φ is the standard normal distribution function.

(a) Write out the log-likelihood function.

(b) Find MLE of $\boldsymbol{\theta}$.

(c) Calculate the Fisher information matrix, $I(\boldsymbol{\theta})$.

Bonus Problem (Optional): (Fisher Information for a Binomial Distribution) For $X \sim \text{Binomial}(n, p)$, derive the Fisher Information $I(p)$.