STAT 7650: Homework 2
(Due: Thursday, 02/18/2025)

Note: Show all your work for the necessary steps to receive full credit.

Please turn in the HW on paper, hand-written and/or typed. For computational problems, return only the relevant parts of the output with comments/annotations. From the code you are using to answer the problems, turn in the relevant output, and the figures (if requested), preferably printed from the output. No need to turn in your code or long lists of generated samples.

Please disclose any us of AI in your solutions. Regardless of you use it or not, make sure you submit your own work, not copy from other source(s). Any suspicion of AI use will result in automatic 0 or substantial point loss in any question.

Q1. Consider the problem of finding the cube root of 4.

(a) What would be the function $f(x)$ whose root corresponds to $\sqrt[3]{4}$?

Approximate the cube root of 4 using

(b) bisection method

(c) Newton's method

(d) secant method

(e) Verify your result by computing $\sqrt[3]{4}$ directly in `R`. Determine the minimum number of iterations to reach the precision of 10^{-10}, and discuss the convergence rate of the methods.

(f) Would **Fisher scoring** be applicable in this context? Why or why not?

Q2. Generate a random sample of size 20 from a Bernoulli distribution, Ber($p = .33$) (use seed=123).

(a) Approximate the **maximum likelihood estimator (MLE)** of p using any of bisection, Newton's, or secant methods.

(b) Repeat part (a) using Fisher scoring method.

(c) Compare your result with the analytical result using the MLE formula from Q2 of HW1.

(d) Determine the minimum number of iterations to reach the precision of 10^{-10}, and discuss the convergence rate of the methods.

Q3. Show that convergence order (or rate of convergence) for the bisection method is 1.

Q4. Can you write each of Newton, Secant, and Fisher Scoring methods as special cases of the Fixed Point Iteration Algorithm. If so write the corresponding $F(x)$ whose fixed point would yield the solution. If not explain why not. (**Hint:** This question is asking whether you can write the updating equation for these algorithms as $x_{t+1} = F(x_t)$ for some function F.)

Q5. Recall that a fixed-point iteration approach seeks to determine a function $F(x)$ such that $f(x) = 0$ if and only if $F(x) = x$. This transforms the problem of finding a root of f into a problem of finding a fixed point of F. In general, the effectiveness of fixed-point iteration is highly dependent on the chosen form of F. For example, consider the root-finding problem for the function

$$f(x) = x + \log x.$$

(a) Show that the following functions are valid choices for F:

$$F_1(x) = \frac{x + e^{-x}}{2}, \quad F_2(x) = e^{-x}, \quad F_3(x) = -\log x.$$

(b) Does fixed-point iteration converge for all choices of F? in part (a). Justify your answer by analyzing the necessary conditions for convergence.

(c) Using numerical experiments, determine which of F_1, F_2, or F_3 converges faster and which converges slower.

(d) **(Optional)** Determine the theoretical convergence rate of each function F_i (i.e., its order of convergence, if it does converge).