SYMMETRIC α-STABLE SUBORDINATORS AND CAUCHY PROBLEMS

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Introduction

In recent years, starting with the articles of Burdzy (1993) and (1994), researchers had interest in iterated processes in which one changes the time parameter with one-dimensional Brownian motion.

To define **iterated Brownian motion** Z_t, due to Burdzy (1993), started at $z \in \mathbb{R}$, let X_t^+, X_t^- and Y_t be three independent one-dimensional Brownian motions, all started at 0. **Two-sided Brownian motion** is defined to be

$$X_t = \begin{cases} X_t^+, & t \geq 0 \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then iterated Brownian motion started at $z \in \mathbb{R}$ is

$$Z_t = z + X(Y_t), \quad t \geq 0.$$

BM versus IBM: This process has many properties analogous to those of Brownian motion; we list a few

(1) Z_t has stationary (but not independent) increments, and is a **self-similar process** of index 1/4.

(2) **Laws of the iterated logarithm (LIL)** holds: usual LIL by Burdzy (1993)

$$\limsup_{t\to\infty} \frac{Z(t)}{t^{1/4}(\log\log(1/t))^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad a.s.$$

Chung-type LIL by Khoshnevisan and Lewis (1996) and Hu et al. (1995).

(3) Khoshnevisan and Lewis (1999) extended results of Burdzy (1994), to develop a **stochastic calculus** for iterated Brownian motion.

(4) In 1998, Burdzy and Khosnevisan showed that IBM can be used to model diffusion in a crack.

(5) Local times of this process was studied by Burdzy and Khosnevisan (1995), Csáki, Csörgö, Földes, and Révész (1996), Shi and Yor (1997), Xiao (1998), and Hu (1999).

(6) Bañuelos and DeBlassie (2006) studied the **distribution of exit place** for iterated Brownian motion in cones.

(7) DeBlassie (2004) studied the lifetime asymptotics of iterated Brownian motion in cones and Bounded domains. Nane (2006), in a series of papers, extended some of the results of DeBlassie. He also studied the lifetime asymtotics of iterated Brownian motion in several unbounded domains(parabola-shaped domains, twisted domains...).

PDE connection

The classical well-known connection of a PDE and a stochastic process is the Brownian motion and heat equation connection. Let $X_t \in \mathbb{R}^n$ be Brownian motion started at x. Then the function

$$u(t,x) = E_x[f(X_t)]$$

solves the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \Delta u(t,x), \qquad t > 0, \quad x \in \mathbb{R}^n \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}^n. \end{aligned}$$

In addition to the above properties of IBM there is an interesting connection between iterated Brownian motion and the **biharmonic operator** Δ^2; the function

$$u(t,x) = E_x[f(Z_t)]$$

solves the Cauchy problem (Allouba and Zheng (2001) and DeBlassie (2004))

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{\Delta f(x)}{\sqrt{2\pi t}} + \frac{1}{2}\Delta^2 u(t,x); \quad (1) \\ u(0,x) &= f(x). \end{aligned}$$

for $t > 0$ and $x \in \mathbb{R}^d$. The non-Markovian property of IBM is reflected by the appearance of the initial function $f(x)$ in the PDE.

Given a Banach space and a bounded continuous semigroup $T(t)$ on that space with generator L_x, it is well known that $p(t,x) = T(t)f(x)$ is the unique solution to the abstract Cauchy problem

$$\frac{\partial}{\partial t}p(t,x) = L_x p(t,x); \quad p(0,x) = f(x) \quad (2)$$

for any f in the domain of L_x see for example Pazy (1983).

For a Markov process X, the family of linear operators

$T(t)f(x) = E_x[f(X(t))] = E[f(X(t))|X(0) = x]$

forms a bounded continuous semigroup on the Banach space $L^1(\mathbb{R}^d)$, and the generator

$$L_x f(x) = \lim_{h \downarrow 0} h^{-1}(T(h)f(x) - f(x))$$

is defined on a dense subset of that space, see for example Hille and Phillips (1957). Then $u(t, x) = T(t)f(x)$ solves the Cauchy problem

$$\frac{\partial}{\partial t} u(t, x) = L_x f(x); \quad u(0, x) = f(x) \qquad (3)$$

for $t > 0$ and $x \in \mathbb{R}^d$.

Allouba and Zheng (2001) show that if we replace the outer process $X(t)$ with a continuous Markov process , the same result holds, except that we replace the Laplacian in the PDE (1) with the generator L_x of the continuous semigroup associated with this Markov process. That is

$$u(t,x) = E_x[f(Z_t)] := E[f(Z_t)|Z_0 = x]$$

solves the Cauchy initial value problem

$$\frac{\partial}{\partial t}u(t,x) = \frac{L_x f(x)}{\sqrt{\pi t}} + {L_x}^2 u(t,x); \quad u(0,x) = f(x) \tag{4}$$

for $t > 0$ and $x \in \mathbb{R}^d$.

Let $q(t,s) = \frac{2}{\sqrt{4\pi t}} \exp\left(-\frac{s^2}{4t}\right)$ be the transition density of one-dimensional Brownian motion.

The essential argument, using integration by parts twice, is that

$$
\begin{aligned}
&\tfrac{\partial}{\partial t}u(t,x) \\
&= \int_0^\infty T(s)f(x)\ \frac{\partial}{\partial t}q(t,s)ds \\
&= \int_0^\infty T(s)f(x)\ \frac{\partial^2}{\partial s^2}q(t,s)ds \\
&= q(t,s)\frac{\partial}{\partial s}\left[T(s)f(x)\right]\Big|_{s=0} \\
&+ \int_0^\infty \frac{\partial^2}{\partial s^2}\left[T(s)f(x)\right]q(t,s)ds \\
&= q(t,0)L_x[T(0)f(x)] + \int_0^\infty L_x^2\left[T(s)f(x)\right]q(t,s)ds \\
&= \frac{1}{\sqrt{\pi t}}L_x f(x) + L_x^2\int_0^\infty T(s)f(x)\ q(t,s)ds
\end{aligned}
$$

Fractional Cauchy problems

Zaslavsky (1994) introduced the fractional kinetic equation

$$\frac{\partial^\beta}{\partial t^\beta}u(t,x) = L_x u(t,x); \quad u(0,x) = f(x) \quad (5)$$

for Hamiltonian chaos, where $0 < \beta < 1$ and L_x is the generator of some continuous Markov process $X_0(t)$ started at $x = 0$. Here $\partial^\beta g(t)/\partial t^\beta$ is the Caputo fractional derivative in time, which can be defined as the inverse Laplace transform of $s^\beta \tilde{g}(s) - s^{\beta-1}g(0)$, with $\tilde{g}(s) = \int_0^\infty e^{-st}g(t)dt$ the usual Laplace transform.

Baeumer and Meerschaert (2001) and Meerschaert and Scheffler (2004) show that the fractional Cauchy problem (5) is related to a certain class of subordinated stochastic processes.

Take D_t to be the stable subordinator, a Lévy process with strictly increasing sample paths such that $E[e^{-sD_t}] = e^{-ts^\beta}$, see for example Bertoin (1996). Define the inverse or hitting time or first passage time process

$$E_t = \inf\{x > 0 : D(x) > t\}. \tag{6}$$

The subordinated process $Z_t = X_0(E_t)$ occurs as the scaling limit of a continuous time random walk (also called a renewal reward process), in which iid random jumps are separated by iid positive waiting times (Meerschaert and Scheffler (2004)).

Theorem 3.1 in Baeumer and Meerschaert (2001) shows that, in the case $p(t,x) = T(t)f(x)$ is a bounded continuous semigroup on a Banach space, the formula

$$\begin{aligned} u(t,x) &= \int_0^\infty p((t/s)^\beta, x) g_\beta(s)\, ds \\ &= \frac{t}{\beta}\int_0^\infty p(x,s) g_\beta(\frac{t}{s^{1/\beta}}) s^{-1/\beta-1} ds \end{aligned}$$

yields a solution to the **fractional Cauchy problem**:

$$\frac{\partial^\beta}{\partial t^\beta} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x) \quad (7)$$

Here $g_\beta(t)$ is the smooth density of the stable subordinator, such that the Laplace transform $\tilde{g}_\beta(s) = \int_0^\infty e^{-st} g_\beta(t)\, dt = e^{-s^\beta}$.

Choose $x \in \mathbb{R}^d$ and let $X(t) = x + X_0(t)$. In the case X_t is a Levy process, essential idea is taking Fourier-Laplace transform and inverting.

The Lévy process $X_0(t)$ has characteristic function

$$E[\exp(ik \cdot X_0(t))] = \exp(t\psi(k))$$

with

$$\begin{aligned} \psi(k) &= ik \cdot a - \frac{1}{2} k \cdot Qk \\ &+ \int_{y \neq 0} \left(e^{ik \cdot y} - 1 - \frac{ik \cdot y}{1 + ||y||^2} \right) \phi(dy), \end{aligned}$$

where $a \in \mathbb{R}^d$, Q is a nonnegative definite matrix, and ϕ is a σ-finite Borel measure on $\mathbb{R}^d$ such that

$$\int_{y \neq 0} \min\{1, ||y||^2\} \phi(dy) < \infty,$$

see for example Theorem 3.1.11 Meerschaert and Scheffler (2001) and Theorem 1.2.14 in Applebaum (2004). Let

$$\hat{f}(k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} f(x)\, dx$$

denote the Fourier transform.

Theorem 3.1 in Baeumer and Meerschaert (2001) shows that $L_x f(x)$ is the inverse Fourier transform of $\psi(k)\widehat{f}(k)$ for all $f \in D(L_x)$, where

$D(L_x) = \{f \in L^1(\mathbb{R}^d) : \ \psi(k)\widehat{f}(k) = \widehat{h}(k) \ \exists \ h \in L^1(\mathbb{R}^d)\}$,
and

$$L_x f(x) = a \cdot \nabla f(x) + \frac{1}{2}\nabla \cdot Q\nabla f(x) + \int_{y \neq 0} \left(f(x+y) - f(x) - \frac{\nabla f(x) \cdot y}{1+y^2} \right) \phi(dy) \tag{8}$$

for all $f \in W^{2,1}(\mathbb{R}^d)$, the Sobolev space of L^1-functions whose first and second partial derivatives are all L^1-functions.

For example, if $X_0(t)$ is spherically symmetric stable then $\psi(k) = -D\|k\|^\alpha$ and

$$L_x = -D(-\Delta)^{\alpha/2},$$

a fractional derivative in space, using the correspondence $k_j \to -i\partial/\partial x_j$ for $1 \le j \le d$.

We will use the following notation for the Laplace, Fourier, and Fourier-Laplace transforms (respectively):

$$\tilde{u}(s,x) = \int_0^\infty e^{-st} u(t,x)dt;$$
$$\hat{u}(t,k) = \int_{\mathbb{R}^d} e^{-ik\cdot x} u(t,x)dx;$$
$$\bar{u}(s,k) = \int_{\mathbb{R}^d} e^{-ik\cdot x} \int_0^\infty e^{-st} u(t,x)dtdx.$$

Taking Fourier transforms on both sides of (5) gives

$$\frac{\partial^\beta \hat{u}(t,k)}{\partial t^\beta} = \psi(k)\hat{u}(t,k)$$

Take Laplace transforms on both sides, using the fact that $s^\beta \tilde{g}(s)) - s^{\beta-1}g(0)$ is the Laplace transform of the Caputo fractional derivative $\partial^\beta g(t)/\partial t^\beta$, to get

$$s^\beta \bar{u}(s,k) - s^{\beta-1}\hat{f}(k) = \psi(k)\bar{u}(s,k)$$

and collect terms to obtain

$$
\begin{aligned}
\bar{u}(s,k) &= \frac{s^{\beta-1}\widehat{f}(k)}{s^{\beta}-\psi(k)} \\
&= s^{\beta-1}\int_0^\infty e^{-t(s^{\beta}-\psi(k))}\widehat{f}(k)dt \\
&= s^{\beta-1}\int_0^\infty e^{-ts^{\beta}}\widehat{T(t)f}(k)dt \\
&= \int_0^\infty -\frac{1}{\beta t}\frac{d}{ds}(e^{-ts^{\beta}})\widehat{T(t)f}(k)dt \\
&= \int_0^\infty -\frac{1}{\beta t}\frac{d}{ds}(\int_0^\infty e^{-t^{1/\beta}sh}g_{\beta}(h)dh)\widehat{T(t)f}(k)dt \\
&= \int_0^\infty \frac{1}{\beta t}\int_0^\infty t^{1/\beta}he^{-t^{1/\beta}sh}g_{\beta}(h)dh\widehat{T(t)f}(k)dt \\
&= \int_0^\infty \int_0^\infty e^{-t's}g_{\beta}(h)T(\widehat{(t'/h)^{\beta}})f(k)dtdh \\
&= F-L(\int_0^\infty p((t/h)^{\beta},x)g_{\beta}(h)\,dh)
\end{aligned}
\tag{9}
$$

Applying change of variables $t' = t^{1/\beta}h$.

Brownian subordiantors and fractional Cauchy problems

Theorem 1 [Baeumer, Meerschaert and Nane (2007)]

Let L_x be the generator of a Markov semigroup $T(t)f(x) = E_x[f(X_t)]$, and take $f \in D(L_x)$ the domain of the generator. Then, both the Cauchy problem

$$\frac{\partial}{\partial t}u(t,x) = \frac{L_x f(x)}{\sqrt{\pi t}} + {L_x}^2 u(t,x); \qquad u(0,x) = f(x), \tag{10}$$

and the fractional Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta}u(t,x) = L_x u(t,x); \qquad u(0,x) = f(x) \tag{11}$$

with $\beta = 1/2$, have the same solution

$$\begin{aligned} u(t,x) &= E_x[f(Z_t)] \qquad (12) \\ &= \frac{2}{\sqrt{4\pi t}} \int_0^\infty T(s)f(x) \exp\left(-\frac{s^2}{4t}\right) ds. \end{aligned}$$

If the outer process is an independent Brownian motion we get the following corollary: For $f \in D(\Delta_x)$, both the Cauchy problem

$$\frac{\partial}{\partial t}u(t,x) = \frac{\Delta_x f(x)}{\sqrt{\pi t}} + \Delta_x^2 u(t,x); \quad u(0,x) = f(x) \tag{13}$$

and the fractional Cauchy problem

$$\frac{\partial^{1/2}}{\partial t^{1/2}}u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x) \tag{14}$$

have the same unique solution given by

$$u(t,x) = \frac{2}{\sqrt{4\pi t}} \int_0^\infty T(s)f(x) \exp\left(-\frac{s^2}{4t}\right) ds, \tag{15}$$

where

$$T(t)f(x) = \int_{\mathbb{R}^d} f(x+y)(4\pi t)^{-d/2} \exp\left(-\frac{\|y\|^2}{4t}\right) dy. \tag{16}$$

The essential idea is showing that E_t and $|Y_t|$ have the same density.

In the case $\beta = 1/2$, we have Example 1.3.19 in Applebaum (2004)

$$g_{1/2}(x) = \frac{1}{\sqrt{4\pi x^3}} \exp\left(-\frac{1}{4x}\right)$$

and then a change of variables shows that

$$\begin{aligned} u(t,x) &= \frac{t}{\beta}\int_0^\infty p(x,s) g_\beta\left(\frac{t}{s^{1/\beta}}\right) s^{-1/\beta - 1} ds \\ &= \int_0^\infty p(x,s) q(t,s) ds \end{aligned} \tag{17}$$

where $p(x,t) = T(t)f(x)$ and

$$\begin{aligned} q(t,s) &= 2t g_{1/2}(t/s^2) s^{-3} \\ &= \frac{2t}{s^3\sqrt{4\pi t^3/s^6}} \exp\left(-\frac{s^2}{4t}\right) \\ &= \frac{2}{\sqrt{4\pi t}} \exp\left(-\frac{s^2}{4t}\right) \end{aligned} \tag{18}$$

is a probability density on $s > 0$ for all $t > 0$. Hence we have that (15) is a solution to the fractional Cauchy problem (11).

Allouba and Zheng (2001) show that the Cauchy problem (10) has solution $u(t,x) = E_x[f(Z_t)]$ where $Z_t = B(|Y_t|)$ is the IBM process. It is not hard to check that the function $q(t,s)$ in (18) is the probability density of the Brownian subordinator $|Y_t|$. Then a simple conditioning argument shows that

$$u(t,x) = E_x[f(Z_t)] = \int_0^\infty q(t,s)p(x,s)ds \quad (19)$$

where $p(x,s) = T(s)f(x) = E_x[f(X_t)]$ is the unique solution to the Cauchy problem (2). Hence both the fractional Cauchy problem (11) and the Cauchy problem (10) have the same solution.

We obtain the following corollary of our theorem

Corollary. **For any continuous Markov process $X(t)$, both the Brownian-time subordinated process $X(|Y_t|)$ and the process $X(E_t)$ subordinated to the inverse $1/2$-stable subordinator have the same one-dimensional distributions. Hence they are both stochastic solutions to the fractional Cauchy problem (5), or equivalently, to the higher order Cauchy problem (4).**

Fourier-Laplace method

The next result is a restatement of Theorem 1 for Lévy semigroups. The proof does not use Theorem 0.1 in Allouba and Zheng (2001), rather it relies on a Laplace-Fourier transform argument.

Theorem 2 [Baeumer, et. al. 2007]

Suppose that $X(t) = x + X_0(t)$ **where** $X_0(t)$ **is a Lévy process starting at zero. If** L_x **is the generator** (8) **of the semigroup** $T(t)f(x) = E_x[(f(X_t))]$ **on** $L^1(\mathbb{R}^d)$**, then for any** $f \in D(L_x)$**, both the initial value problem** (4)**, and the fractional Cauchy problem** (5) **with** $\beta = 1/2$**, have the same unique solution given by** (15)**.**

Take Fourier transforms on both sides of (4) to get

$$\frac{\partial \widehat{u}(t,k)}{\partial t} = \frac{1}{\sqrt{\pi t}}\psi(k)\widehat{f}(k) + \psi(k)^2\widehat{u}(t,k)$$

using the fact that $\psi(k)\widehat{f}(k)$ is the Fourier transform of $L_x f(x)$. Then take Laplace transforms on both sides to get

$$s\bar{u}(s,k) - \widehat{u}(t=0,k) = s^{-1/2}\psi(k)\widehat{f}(k) + \psi(k)^2\bar{u}(s,k),$$

using the well-known Laplace transform formula

$$\int_0^\infty \frac{t^{-\beta}}{\Gamma(1-\beta)} e^{-st} dt = s^{\beta-1}$$

for $\beta < 1$. Since $\hat{u}(t=0,k) = \hat{f}(k)$, collecting like terms yields

$$\bar{u}(s,k) = \frac{(1 + s^{-1/2}\psi(k))\hat{f}(k)}{s - \psi(k)^2} \tag{20}$$

for $s > 0$ sufficiently large.

On the other hand, taking Fourier transforms on both sides of (5) with $\beta = 1/2$ gives

$$\frac{\partial^{1/2}\hat{u}(t,k)}{\partial t^{1/2}} = \psi(k)\hat{u}(t,k)$$

Take Laplace transforms on both sides, using the fact that $s^\beta \tilde{g}(s)) - s^{\beta-1} g(0)$ is the Laplace transform of the Caputo fractional derivative $\partial^\beta g(t)/\partial t^\beta$, to get

$$s^{1/2}\bar{u}(s,k) - s^{-1/2}\hat{f}(k) = \psi(k)\bar{u}(s,k)$$

and collect terms to obtain

$$\begin{aligned}\bar{u}(s,k) &= \frac{s^{-1/2}\widehat{f}(k)}{s^{1/2}-\psi(k)} \\ &= \frac{s^{-1/2}\widehat{f}(k)}{s^{1/2}-\psi(k)} \cdot \frac{s^{1/2}+\psi(k)}{s^{1/2}+\psi(k)} \qquad (21) \\ &= \frac{(1+s^{-1/2}\psi(k))\widehat{f}(k)}{s-\psi(k)^2}\end{aligned}$$

which agrees with (20). For any fixed $k \in \mathbb{R}^d$, the two formulas are well-defined and equal for all $s > 0$ sufficiently large.

An easy extension of the argument for Theorem 2 shows that, under the same conditions, for any $n = 2, 3, 4, \ldots$ both the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = \sum_{j=1}^{n-1} \frac{t^{1-j/n}}{\Gamma(j/n)} L_x^j f(x) + L_x^n u(t,x);$$
$$u(0,x) = f(x) \tag{22}$$

and the fractional Cauchy problem:

$$\frac{\partial^{1/n}}{\partial t^{1/n}} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x), \tag{23}$$

have the same unique solution given by

$$u(t,x) = \int_0^\infty p((t/s)^\beta, x) g_\beta(s)\, ds$$

with $\beta = 1/n$. Hence the process $Z_t = X(E_t)$ is also the stochastic solution to this higher order Cauchy problem.

Other subordinators

α-time process is a Markov process subordinated to the absolute value of an independent one-dimensional symmetric α-stable process:

$Z_t = B(|S_t|)$, where B_t is a Markov process and S_t is an independent symmetric α-stable process both started at 0.

This process is self similar with index $1/2\alpha$ when the outer process X is a Brownian motion. In this case Nane (2006) defined the Local time of this process and obtained Laws of the iterated logarithm for the local time for large time.

PDE-connection:

Theorem 3 [Nane 2005]

Let $T(s)f(x) = E[f(X^x(s))]$ **be the semi-group of the continuous Markov process** $X^x(t)$ **and let** L_x **be its generator. Let** $\alpha = 1$**. Let** f **be a bounded measurable function in the domain of** $\mathcal{A}$**, with** $D_{ij}f$ **bounded and Hölder continuous for all** $1 \leq i,\ j \leq n$**. Then** $u(t,x) = E[f(Z_t^x)]$ **solves**

$$\begin{aligned} \frac{\partial^2}{\partial t^2}u(t,x) &= -\frac{2L_x f(x)}{\pi t} - L_x^2 u(t,x); \\ u(0,x) &= f(x). \end{aligned}$$

In particular, if $X^x(t)$ is Brownian motion started at x and Δ is the standard Laplacian, then u solves

$$\begin{aligned} \frac{\partial^2}{\partial t^2}u(t,x) &= -\frac{2\Delta f(x)}{\pi t} - \Delta^2 u(t,x); \\ u(0,x) &= f(x). \end{aligned}$$

We use the representation

$$u(t,x) = E[f(Z_t^x)] = 2\int_0^\infty p_t^1(0,s)T(s)f(x)ds,$$

where $p_t^1(0,s) = \frac{t}{\pi(s^2+t^2)}$ is the transition density of the Cauchy process on $\mathbb{R}$. Using dominated convergence and the fact that $p_t^1(0,s)$ satisfy

$$(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2})p_t^1(0,s) = 0,$$

we obtain

$$\begin{aligned} \frac{\partial^2}{\partial t^2}E[f(Z_C^x(t))] &= 2\int_0^\infty \frac{\partial^2}{\partial t^2}p_t^1(0,s)T(s)f(x)ds \\ &= 2\int_0^\infty -\frac{\partial^2}{\partial s^2}p_t^1(0,s)T(s)f(x)ds. \\ &= 2\int_0^\infty \frac{\partial}{\partial s}p_t^1(0,s)\frac{\partial}{\partial s}T(s)f(x)ds \\ &= -2p_t^1(0,0)L_x f(x) \\ &\quad -2\int_0^\infty p_t^1(0,s)L_x^2 T(s)f(x)ds. \end{aligned}$$

Taking the application of L_x^2 outside the integral using conditions on f and $D_{ij}f$ we get

$$\frac{\partial^2}{\partial t^2}u(t,x) = -2p_t^1(0,0)L_x f(x) \ - \ L_x^2 u(t,x).$$

For $\alpha = l/m \neq 1$ **rational:** the PDE is more complicated since kernels of symmetric α-stable processes satisfy a higher order PDE:

$$(\frac{\partial^2}{\partial s^2})^l + (-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}})p_t^\alpha(0,s) = 0.$$

We also have to assume that we can integrate under the integral as much as we need in the case where the outer process is BM (or in general we can take the operator out of the integral). This is valid for $\alpha = 1/m$, $m = 2, 3, \cdots$ by a Lemma in Nane (2005).

Theorem 4 [Nane (2005)]

Let $\alpha \in (0,2)$ **be rational** $\alpha = l/m$**, where** l **and** m **are relatively prime. Let** $\mathcal{T}_s f(x) = E[f(X^x(s))]$ **be the semigroup of the continuous Markov process** $X^x(t)$ **and let** L_x **be its generator. Let** f **be a bounded measurable function in the domain of** L_x**, with** $D^\gamma f$ **bounded and Hölder continuous for all multi index** γ **such that** $|\gamma| = 2l$**. Then** $u(t,x) = E[f(Z_t^x)]$ **solves**

$$
\begin{aligned}
&(-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}}u(t,x) \\
&\qquad = -2\sum_{i=1}^{l}\left(\frac{\partial^{2l-2i}}{\partial s^{2l-2i}}p_t^{\alpha}(0,s)|_{s=0}\right)L_x^{2i-1}f(x) \\
&\qquad\quad - L_x^{2l}u(t,x); \\
&\qquad u(0,x) = f(x).
\end{aligned}
$$

Open Problems

Question 1. For $\beta = 1/2$, the inverse stable subordinator process E_t (E_t is also the Local time process of one-dimensional Brownian motion at 0) and the process $|Y_t|$, where Y_t is a one-dimensional Brownian motion have the same transition density. Is there a similar correspondence between E_t for $\beta \neq 1/2$ and other symmetric α-stable process Y_t for $1 < \alpha < 2$.

Question 2. Looking at the governing PDE for subordinators other than Brownian motion, are there any fractional in time PDE which has the same solution as the higher order pde?

Question 3. Are there PDE connections of the iterated processes in bounded domain as the PDE connection of Brownian motion in bounded domains?