# ITERATED BROWNIAN MOTION AND A RELATED CLASS OF PROCESSES

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# OUTLINE

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- Iterated processes in unbounded domains
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## INTRODUCTION

In recent years, starting with the articles of Burdzy (1993) and (1994), researchers had interest in iterated processes in which one changes the time parameter with one-dimensional Brownian motion.

To define **iterated Brownian motion** $Z_t$, due to Burdzy (1993), started at $z \in \mathbb{R}$, let $X_t^+$, $X_t^-$ and $Y_t$ be three independent one-dimensional Brownian motions, all started at $0$. **Two-sided Brownian motion** is defined to be

$$X_t = \begin{cases} X_t^+, & t \geq 0 \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then iterated Brownian motion started at $z \in \mathbb{R}$ is

$$Z_t = z + X_{(Y_t)}, \quad t \geq 0.$$

**BM versus IBM:** This process has many properties analogous to those of Brownian motion; we list a few

(1) $Z_t$ has stationary (but not independent) increments, and is a **self-similar process** of index $1/4$.

(2) **Laws of the iterated logarithm (LIL)** holds: usual LIL by Burdzy (1993)

$$\limsup_{t\to\infty} \frac{Z(t)}{t^{1/4}(\log\log(1/t))^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad a.s.$$

Chung-type LIL by Khoshnevisan and Lewis (1996) and Hu et al. (1995).

(3) Khoshnevisan and Lewis (1999) extended results of Burdzy (1994), to develop a **stochastic calculus** for iterated Brownian motion.

(4) In 1998, Burdzy and Khoshnevisan showed that IBM can be used to model diffusion in a crack.

(5) Local time of this process was studied by Burdzy and Khosnevisan (1995), Csáki, Csörgö, Földes, and Révész (1996), Shi and Yor (1997), Xiao (1998), and Hu (1999).

(6) Khoshnevisan and Lewis (1996) established the modulus of continuity for iterated Brownian motion: with probability one

$$\lim_{\delta\to 0}\sup_{0\le s,t\le 1}\sup_{0\le|s-t|\le\delta}\frac{|Z(s)-Z(t)|}{\delta^{1/4}(\log(1/\delta))^{3/4}}=1.$$

(7) Bañuelos and DeBlassie (2006) studied the **distribution of exit place** for iterated Brownian motion in cones.

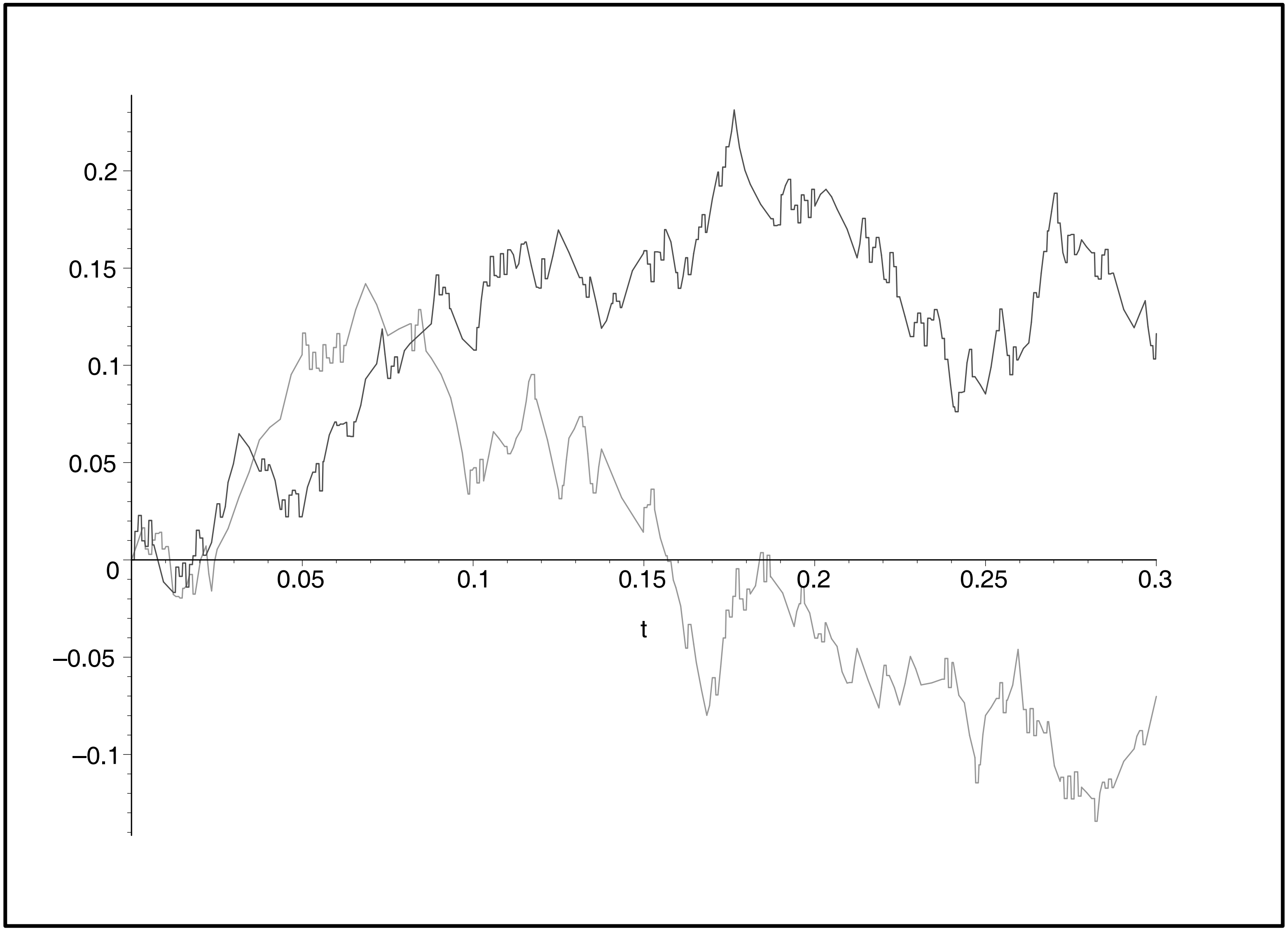


Figure 1: Simulations of two Brownian motions

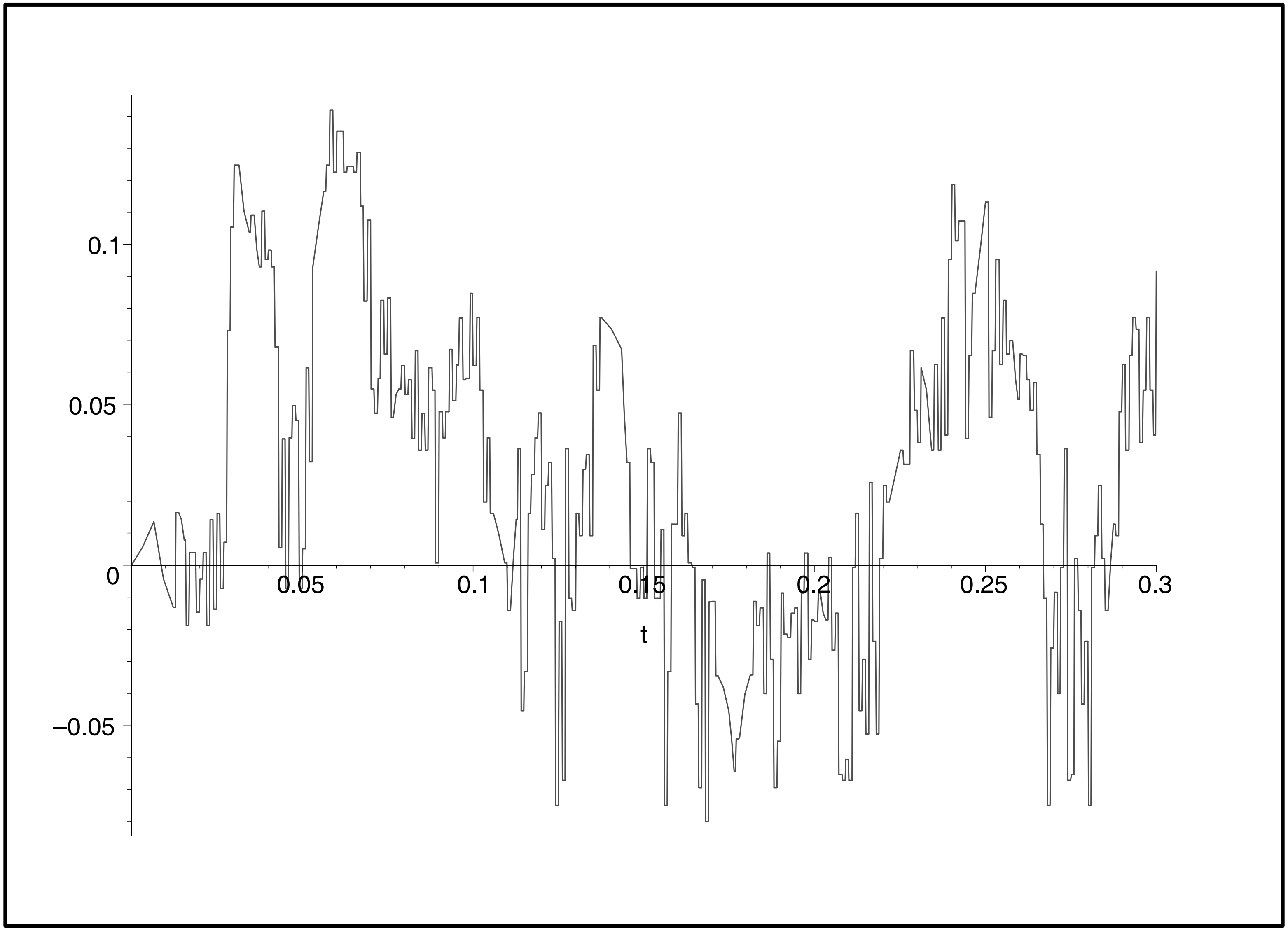




Figure 2: Simulation of IBM $Z_t^1 = X(|Y_t|)$

# ITERATED PROCESSES IN UNBOUNDED DOMAINS

Let $D$ be a domain in $\mathbb{R}^n$. Let

$$\tau_D(Z) = \inf\{t \geq 0 : \ Z_t \notin D\}$$

be the first exit time of $Z_t$ from $D$. Write

$$\tau_D^{\pm}(z) = \inf\{t \geq 0 : \ \ X_t^{\pm} + z \notin D\},$$

and if $I \subset \mathbb{R}$ is an open interval, write

$$\eta_I = \eta(I) = \inf\{t \geq 0 : \ \ Y_t \notin I\}.$$

By continuity of the paths of $Z_t = z + X(Y_t)$ (for $f$ the pdf of $\tau_D^{\pm}(z)$)

$$
\begin{aligned}
P_z[\tau_D(Z) > t] & \\
&= P_z[Z_s \in D \text{ for all } s \le t] \\
&= P[z + X^+(0 \vee Y_s) \in D \text{ and} \\
&\quad z + X^-(0 \vee (-Y_s)) \in D \text{ for all } s \le t] \\
&= P[\tau_D^+(z) > 0 \vee Y_s \text{ and } \tau_D^-(z) > 0 \vee (-Y_s) \\
&\quad \text{for all } s \le t] \\
&= P[-\tau_D^-(z) < Y_s < \tau_D^+(z) \text{for all } s \le t] \\
&= P[\eta(-\tau_D^-(z), \tau_D^+(z)) > t], \\
&= \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f(v) dv du.
\end{aligned}
$$

Let $\tau_D$ be the first exit time of the Brownian motion $X_t$ from $D$. In the case of Brownian motion in *generalized cones*, this has been done by several people including Bañuelos and Smits (1997), Burkholder (1977) and DeBlassie (1987): for $x \in D$,

$$P_x[\tau_D > t] \sim C(x)t^{-p(D)}, \text{ as } t \to \infty.$$

When $D$ is a generalized cone, using the results of Bañuelos and Smits, DeBlassie obtained;

**Theorem 1 (DeBlassie (2004))** *For $z \in D$, as $t \to \infty$,*

$$P_z[\tau_D(Z) > t] \approx \begin{cases} t^{-p(D)}, & p(D) < 1 \\ t^{-1}\ln t, & p(D) = 1 \\ t^{-(p(D)+1)/2}, & p(D) > 1. \end{cases}$$

*Here $f \approx g$ means that for some positive $C_1$ and $C_2$, $C_1 \leq f/g \leq C_2$.*

For *parabola-shaped domains* the study of exit time asymptotics for Brownian motion was initiated by Bañuelos, DeBlassie and Smits.

**Theorem 2 (Bañuelos, et al. (2001))** *Let*

$$\mathcal{P} = \{(x, y) : x > 0, |y| < \sqrt{x}\}.$$

*Then for* $z \in \mathcal{P}$*,*

$$\log\ P_z[\tau_{\mathcal{P}} > t] \approx -t^{\frac{1}{3}}$$

Subsequently, Lifshits and Shi found that the above *limit exists* for parabola-shaped domains $P_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$, $0 < \alpha < 1$ and $A > 0$ in any dimension;

**Theorem 3 (Lifshits and Shi (2002))** *For* $z \in P_\alpha$,

$$\lim_{t\to\infty} t^{-\left(\frac{1-\alpha}{1+\alpha}\right)} \log P_z[\tau_{P_\alpha} > t] = -l, \tag{1}$$

*where*

$$l = \left(\frac{1+\alpha}{\alpha}\right) \left( L \frac{\Gamma^2\left(\frac{1-\alpha}{2\alpha}\right)}{\Gamma^2\left(\frac{1}{2\alpha}\right)} \right)^{\frac{\alpha}{(\alpha+1)}}. \tag{2}$$

*where*

$$L = \frac{\pi \jmath_{(n-3)/2}^{2/\alpha}}{A^2 2^{(3\alpha+1)/\alpha}((1-\alpha)/\alpha)^{(1-\alpha)/\alpha}}.$$

Here $\jmath_{(n-3)/2}$ denotes the smallest positive zero of the Bessel function $J_{(n-3)/2}$ and $\Gamma$ is the Gamma function.

By integration by parts $P_z[\tau_D(Z) > t]$ equals to

$$\int_0^\infty \int_0^\infty \left( \frac{\partial}{\partial u}\frac{\partial}{\partial v} P_0[\eta_{(-u,v)} > t] \right) .P[\tau_D(z) > u] P[\tau_D(z) > v] dv du.$$

**Theorem 4 (Nane (2006))** *Let* $0 < \alpha < 1$, $A > 0$ *and let*

$$P_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}.$$

*Then for* $z \in P_\alpha$,

$$\lim_{t\to\infty} t^{-(\frac{1-\alpha}{3+\alpha})} \log P_z[\tau_{P_\alpha}(Z) > t] = -C_\alpha,$$

*where for* $l$ *as in the limit given by (2)*

$$C_\alpha = (\frac{3+\alpha}{2+2\alpha})(\frac{1+\alpha}{1-\alpha})^{(\frac{1-\alpha}{3+\alpha})} \pi^{(\frac{2-2\alpha}{3+\alpha})} l^{(\frac{2+2\alpha}{3+\alpha})}.$$

In particular, for a **planar iterated Brownian motion in a parabola**, the limit $l = 3\pi^2/8$ in equation (2). Then from Theorem 4 for $z \in \mathcal{P}$,

$$\lim_{t\to\infty} t^{-\frac{1}{7}} \log P_z[\tau_{\mathcal{P}}(Z) > t] = -\frac{7\pi^2}{2^{25/7}}.$$

## ITERATED PROCESSES IN BOUNDED DOMAINS

For many bounded domains $D \subset \mathbb{R}^n$ the asymptotics of $P_z[\tau_D > t]$ is well-known. For $z \in D$,

$$\lim_{t\to\infty} e^{\lambda_D t} P_z[\tau_D > t] = \psi(z) \int_D \psi(y)dy, \qquad (3)$$

where $\lambda_D$ is the first eigenvalue of $\frac{1}{2}\Delta$ with Dirichlet boundary conditions and $\psi$ is its corresponding eigenfunction.

DeBlassie proved the following result for iterated Brownian motion in bounded domains;

**Theorem 5 (DeBlassie (2004))** *For* $z \in D$*,*

$$\lim_{t\to\infty} t^{-1/3} \log P_z[\tau_D(Z) > t] = -\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}. \qquad (4)$$

We have the following theorem which improves the limit in (4).

**Theorem 6 (Nane (2006))** *Let $D \subset \mathbb{R}^n$ be a bounded domain for which (3) holds point-wise and let $\lambda_D$ and $\psi$ be as above. Then for $z \in D$,*

$$\lim_{t\to\infty} t^{-1/2} \exp\left(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}\right) P_z[\tau_D(Z) > t]$$

$$= \frac{\lambda_D 2^{7/2}}{\sqrt{3\pi}} \left(\psi(z)\int_D \psi(y)dy\right)^2 .$$

## Ingredients of the proof of Theorem 6

It turns out that the integral over the set $A$ is the dominant one: $K > 0$ and $M > 0$ define $A$ as

$$A = \left\{ (u,v): \quad K \le u \le \frac{1}{2}\sqrt{\frac{t}{M}},\ u \le v \le \sqrt{\frac{t}{M}} - u \right\}.$$

As $t \to \infty$, uniformly for $x \in (0,1)$,

$$\begin{aligned} P_x[\eta_{(0,1)} > t] &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}e^{-2(n+1)^2\pi^2 t/2}\sin(2n+1)\pi x \\ &\sim \frac{4}{\pi}e^{-\pi^2 t/2}\sin\pi x. \end{aligned}$$

We use **Laplace transform method for integrals** (de Bruijn (1958)):

Let $h$ and $f$ be continuous functions on $\mathbb{R}$. Suppose $f$ is non-positive and has a global max at $x_0$, $f'(x_0) = 0$, $f''(x_0) < 0$ and $h(x_0) \neq 0$ and

$$\int_0^\infty h(x) \exp(\lambda f(x)) < \infty$$

for all $\lambda > 0$. Then as $\lambda \to \infty$,

$$\int_0^\infty h(x) \exp(\lambda f(x)) dx$$

$$\sim \; h(x_0) \exp(\lambda f(x_0)) \sqrt{\frac{2\pi}{\lambda |f''(x_0)|}}.$$

$$
\begin{aligned}
P_z[\tau_D(Z) > t] &= \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f(v) dv du \\
&= \int_0^\infty \int_0^\infty P_{\frac{u}{u+v}}[\eta_{(0,1)} > \frac{t}{(u+v)^2}] f(u) f(v) dv du \\
&\geq C^1 \int_K^{\frac{1}{2}\sqrt{t/M}} \int_u^{\sqrt{t/M}-u} \sin\left(\frac{\pi u}{(u+v)}\right) \\
&\qquad . \exp(-\frac{\pi^2 t}{2(u+v)^2}) \exp(-\lambda_D(u+v)) dv du,
\end{aligned}
$$

where $C^1 = C^1(z) = 2(4/\pi)A(z)^2(1-\epsilon)^3$.

Changing the variables $x = u+v, z = u$ the integral is

$$
= C^1 \int_K^{\frac{1}{2}\sqrt{t/M}} \int_{2z}^{\sqrt{t/M}} \sin\left(\frac{\pi z}{x}\right) \exp(-\frac{\pi^2 t}{2x^2}) \exp(-\lambda_D x) dx dz,
$$

and reversing the order of integration

$$= C^1 \int_{2K}^{\sqrt{t/M}} \int_{K}^{\frac{1}{2}x} \sin\left(\frac{\pi z}{x}\right) \exp(-\frac{\pi^2 t}{2x^2}). \exp(-\lambda_D x) dz dx$$

$$= C^1/\pi \int_{2K}^{\sqrt{t/M}} x \cos\left(\frac{\pi K}{x}\right) \exp(-\frac{\pi^2 t}{2x^2}) \exp(-\lambda_D x) dx$$

By Laplace transform method, after making the change of variables $x = (atb^{-1})^{1/3}u$, for $a = \pi^2/2$, $b = \lambda_D$. As $t \to \infty$,

$$\int_0^\infty x \cos\left(\frac{\pi K}{x}\right) \exp(-\frac{\pi^2 t}{2x^2} - \lambda_D x) dx$$

$$= \int_0^\infty (atb^{-1})^{1/3} u \cos\left(\frac{\pi K}{(atb^{-1})^{1/3} u}\right)$$

$$. \exp\left(-a^{1/3} b^{2/3} t^{1/3} (\frac{1}{u^2} + u)\right) (atb^{-1})^{1/3} du$$

$$\sim 2\sqrt{\frac{\pi}{3}} (\frac{\pi^2}{2})^{1/2} \lambda_D^{-1} t^{1/2} \exp(-\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}).$$

Above $x_0$ in the Laplace Transform method is $2^{1/3}$.

## ISOPERIMETRIC-TYPE INEQUALITIES

Let $D \subset \mathbb{R}^n$ be a **domain of finite volume**, and denote by $D^*$ the ball in $\mathbb{R}^n$ centered at the origin with same volume as $D$. **The class of quantities related to the Dirichlet Laplacian in $D$ which are maximized or minimized by the corresponding quantities for $D^*$ are often called generalized isoperimetric-type inequalities (C. Bandle (1980)).**

Probabilistically generalized isoperimetric-type inequalities read as

$$P_z[\tau_D > t] \;\leq\; P_0[\tau_{D^*} > t] \tag{5}$$

for all $z \in D$ and all $t > 0$, where $\tau_D$ is the first exit time of Brownian motion from the domain $D$ and $P_z$ is the associated probability measure when this process starts at $z$.

**Theorem 7 (Nane (2008))** *Let $D \subset \mathbb{R}^n$ be an open set of finite volume. Then*

$$P_z[\tau_D(Z) > t] \;\leq\; P_0[\tau_{D^*}(Z) > t] \tag{6}$$

*for all $z \in D$ and all $t > 0$.*

## Proof of Theorem 7

The idea of the proof is to use **integration by parts and the corresponding generalized isoperimetric-type inequalities for Brownian motion.** Let $f^*$ denote the probability density of $\tau_{D^*}$.

$$G_x(u,v,t) = \left(\frac{\partial}{\partial x} P_0[\eta_{(-u,v)} > t]\right), \;\; \text{for } x = u, v.$$

$P_z[\tau_D(Z) > t]$ equals

$$\begin{aligned} & \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f(v) dv du. \\ = & \int_0^\infty \int_0^\infty G_v(u,v,t) P[\tau_D(z) > v] f(u) dv du \\ \leq & \int_0^\infty \int_0^\infty G_v(u,v,t) P[\tau_{D^*}(0) > v] f(u) dv du \\ = & \int_0^\infty \int_0^\infty P_0[\eta_{(-u,v)} > t] f(u) f^*(v) dv du \\ = & \int_0^\infty \int_0^\infty G_u(u,v,t) P[\tau_D(z) > u] f^*(v) du dv \\ \leq & \int_0^\infty \int_0^\infty G_u(u,v,t) P[\tau_{D^*}(0) > u] f^*(v) du dv \\ = & P_0[\tau_{D^*}(Z) > t] \end{aligned}$$

## LARGE DEVIATIONS FOR A RELATED CLASS OF PROCESSES

Self-similar processes arise naturally in limit theorems of random walks and other stochastic processes, and they have been applied to model various phenomena in a wide range of scientific areas including telecommunications, turbulence, image processing and finance.

The most important example of self-similar processes is *fractional Brownian motion (fBm)* which is a centered Gaussian process $W^H = \{W^H(t), t \in \mathbb{R}\}$ with $W^H(0) = 0$ and covariance function

$$\mathbb{E}\big(W^H(s)W^H(t)\big) = \frac{1}{2}\Big(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\Big), \quad (7)$$

where $H \in (0,1)$ is a constant. When $H = 1/2$, $W^H$ is a two-sided Brownian motion.

A Lévy process $S = \{S_t, t \geq 0\}$ with values in $\mathbb{R}$ is called *strictly stable of index $\alpha \in (0,2]$* if its characteristic function is given by

$$\mathbb{E}\big[\exp(i\xi S_t)\big] = \exp\left(-t|\xi|^{\alpha}\frac{1 + i\nu\,\mathrm{sgn}(\xi)\tan(\frac{\pi\alpha}{2})}{\chi}\right), \quad (8)$$

where $-1 \leq \nu \leq 1$ and $\chi > 0$ are constants. When $\alpha = 2$ and $\chi = 2$, $S$ is Brownian motion.

For any Borel set $I \subseteq \mathbb{R}$, the occupation measure of $S$ on $I$ is defined by

$$\mu_I(A) = \lambda_1\{t \in I : \ S_t \in A\} \quad (9)$$

for all Borel sets $A \subseteq \mathbb{R}$, where $\lambda_1$ is the one-dimensional Lebesgue measure. If $\mu_I$ is absolutely continuous with respect to the Lebesgue measure $\lambda_1$ on $\mathbb{R}$, we say that $S$ has a local time on $I$ and define its *local time $L(x, I)$* to be the Radon-Nikodým derivative of $\mu_I$ with respect to $\lambda_1$, i.e.,

$$L(x, I) = \frac{d\mu_I}{d\lambda_1}(x), \qquad \forall x \in \mathbb{R}.$$

In the above, $x$ is the so-called *space variable*, and $I$ is the *time variable* of the local time. If $I = [0, t]$, we will write $L(x, I)$ as $L(x, t)$. Moreover, if $x = 0$ then we will simply write $L(0, t)$ as $L_t$.

It is well-known (see, e.g. Bertoin (1996)) that a strictly stable Lévy process $S$ has a local time if and only if $\alpha \in (1, 2]$.

## RELATED CLASS OF PROCESSES

Let $W^H = \{W^H(t), t \in \mathbb{R}\}$ be a *fractional Brownian motion* of Hurst index $H \in (0,1)$ with values in $\mathbb{R}$. Let $S = \{S_t, t \geq 0\}$ be a real-valued, *strictly stable Lévy process* of index $1 < \alpha \leq 2$. We assume that $S$ is independent of $W^H$. Let $L = \{L_t, t \geq 0\}$ be the *local time* process at zero of $S$.

Let $Z^H = \{Z^H(t), t \geq 0\}$ be a real-valued stochastic process defined by

$$Z^H(t) = W^H(L_t)$$

for all $t \geq 0$. This iterated process will be called a *local time fractional Brownian motion*

Some properties:

(1) Since the sample functions of $W^H$ and $L$ are a.s. continuous, the local time fractional Brownian motion $Z^H$ also has continuous sample functions.

(2) Moreover, by using the facts that $W^H$ is self-similar with index $H$ and $L$ is self-similar with index $1 - 1/\alpha$, one can readily verify that $Z^H$ is self-similar with index $H(1 - 1/\alpha)$. However, $Z^H$ is non-Gaussian, non-Markovian and does not have stationary increments.

(3) The local time Brownian motion $Z^{1/2}$ emerges as the scaling limit of a continuous time random walk with heavy-tailed waiting times between jumps Meerschaert et al. (2004).

**Theorem 8 (Baeumer, Meerschaert and Nane (2007))** *For* $H = 1/2$ *and* $\alpha = 2$,

$$Z_t^{1/2} = W^{1/2}(L_t)$$

*and the process*

$$W^{1/2}(Y_t) = Z_t$$

*(IBM) subordinated to another one-dimensional Brownian motion* $Y_t$ *independent of* $W^{1/2}$ *have the same one-dimensional distributions.*

**Theorem 9 (Meerschaert, Nane and Xiao (2007))** *Let* $Z^H = \{Z^H(t), t \geq 0\}$ *be a local time fractional Brownian motion with values in* $\mathbb{R}$ *and* $2H < \alpha$. *Then for every Borel set* $D \subseteq \mathbb{R}$,

$$\limsup_{t\to\infty} t^{-\frac{2H(\alpha-1)}{\alpha-2H}} \log \mathbb{P}\Big\{t^{-\frac{2H(\alpha-1)}{\alpha-2H}} Z^H(t) \in D\Big\} \leq -\inf_{x\in\overline{D}} \Lambda_1^*(x) \tag{10}$$

*and*

$$\liminf_{t\to\infty} t^{-\frac{2H(\alpha-1)}{\alpha-2H}} \log \mathbb{P}\Big\{t^{-\frac{2H(\alpha-1)}{\alpha-2H}} Z^H(t) \in D\Big\} \geq -\inf_{x\in D^\circ} \Lambda_1^*(x), \tag{11}$$

*where* $\overline{D}$ *and* $D^\circ$ *denote respectively the closure and interior of* $D$ *and*

$$\Lambda_1^*(x) = \frac{\alpha+2H}{2\alpha}\left(\frac{\alpha-2H}{2\alpha B_1}\right)^{\frac{\alpha-2H}{\alpha+2H}} x^{\frac{2\alpha}{\alpha+2H}}, \qquad \forall\, x \in \mathbb{R}. \tag{12}$$

*In the above, $B_1 = B_1(H, \alpha, \chi, \nu)$ is the positive constant which we can calculate explicitly and where $\nu \in [-1, 1]$ and $\chi > 0$ are the parameters of the stable Lévy process $X$ defined in (8).*

In the terminology of (Dembo and Zeitouni (1998)), Theorem 9 states that the pair

$$\left(t^{-\frac{2H(\alpha-1)}{\alpha-2H}} Z^H(t), t^{\frac{2H(\alpha-1)}{\alpha-2H}}\right)$$

satisfies a large deviation principle with good rate function $\Lambda_1^*$.

Letting $D = [x, \infty)$, we derive from Theorem 9 and the self-similarity of $Z^H$ the asymptotic tail probability $\mathbb{P}\{Z^H(1) \geq x\}$ as $x \to \infty$.

The following theorem is more general because it holds for all $H \in (0, 1)$ and $\alpha \in (1, 2]$.

**Theorem 10 (Meerschaert, Nane and Xiao (2007))** *Let $Z^H = \{Z^H(t), t \geq 0\}$ be a local time fractional Brownian motion with values in $\mathbb{R}$. Then for any $0 \leq a \leq b < \infty$,*

$$\lim_{x \to \infty} \frac{\log \mathbb{P}\left\{\left|Z^H(b) - Z^H(a)\right| > x\right\}}{x^{\frac{2\alpha}{\alpha+2H}}} = -B_2, \qquad (13)$$

*where $B_2 = B_2(H, \alpha, \chi, \nu)$ is the positive constant defined by*

$$B_2 = \frac{\alpha + 2H}{2\alpha} \left(\frac{H\, A_1^{\alpha}}{\left(1 - \frac{1}{\alpha}\right)^{\alpha - 1}}\right)^{-\frac{2H}{\alpha+2H}} \left(b - a\right)^{-\frac{2H(\alpha-1)}{\alpha+2H}}. \qquad (14)$$

In order to prove Theorems 9 and 10, we first study the analytic properties of the moment generating functions of $Z^H(t)$ and $|Z^H(b) - Z^H(a)|$. This is done by calculating the moments of $Z^H(t)$ and $|Z^H(b) - Z^H(a)|$ for $0 \leq a \leq b$ directly and by using a theorem of Valiron (1949). Then Theorems 9 and 10 follow respectively from the Gärtner-Ellis Theorem and a result of Davies (1976).

## APPLICATIONS

**Theorem 11 (Meerschaert, Nane and Xiao (2007))** *Let* $Z^H = \{Z^H(t), t \geq 0\}$ *be a local time fractional Brownian motion with values in* $\mathbb{R}$*. Then there exists a finite constant* $A_9 > 0$ *such that for all constants* $0 \leq a < b < \infty$*, we have*

$$\limsup_{h \downarrow 0} \sup_{a \leq t \leq b-h} \sup_{0 \leq s \leq h} \frac{\left|Z^H(t+s) - Z^H(t)\right|}{h^{H(\alpha-1)/\alpha}\big(\log 1/h\big)^{(\alpha+2H)/(2\alpha)}} \leq A_9 \quad \text{a.s.} \tag{15}$$

**Theorem 12 (Meerschaert, Nane and Xiao (2007))** *Let* $Z^H = \{Z^H(t), t \geq 0\}$ *be a local time fractional Brownian motion with values in* $\mathbb{R}$*. Then almost surely,*

$$\limsup_{t\to\infty} \frac{\max_{0\le s\le t} \left|Z^H(s)\right|}{t^{H(\alpha-1)/\alpha}\big(\log\log t\big)^{(\alpha+2H)/(2\alpha)}} \leq A_8^{-(\alpha+2H)/(2\alpha)}. \tag{16}$$

*In the above,* $A_8$ *is a certain constant that depends on* $H$ *and the parameters of* $S$*.*

Csáki, Földes and Révész (1997) obtained a Strassen type law of the iterated logarithm (LIL) for $Z(t) = W(L_t)$ when $L_t$ is the local time at zero of a **symmetric** stable Lévy process. Our theorem extends partially their result to $Z^H$.

## FUTURE RESEARCH PLANS

- Weak convergence to *local time fractional Brownian motion*. Equivalently, domain of attraction for this process.
- Correlation structure of *local time fractional Brownian motion*.
- Proving lower bounds in Theorems 11 and 12.
- Potential theory for iterated processes. Hausdorff measure and dimension results for iterated processes.

THANK YOU

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