MH 7500 THEOREMS

Definition. A *topological space* is an ordered pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a collection of subsets of X such that

(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
(ii) $U \cap V \in \mathcal{T}$ whenever $U, V \in \mathcal{T}$;
(iii) $\cup \mathcal{U} \in \mathcal{T}$ whenever $\mathcal{U} \subset \mathcal{T}$.

$\mathcal{T}$ is called a *topology* on the set X. Elements of $\mathcal{T}$ are called *open sets*.

Remark. To put it in other words, condition (ii) says that pairwise (and hence finite) intersections of open sets are open, and (iii) says that any union of open sets is open.

Convention. We will often write X *is a topological space* or just X *is a space* to mean that X is a set with some fixed but unspecified topology.

Definition. If X is a space, and $A \subset X$, then A is *closed* iff $X \setminus A$ is open.

Lemma 1. *If X is any set, and $\mathcal{A}$ any collection of subsets of X, then*

(i) $X \setminus \cup \mathcal{A} = \bigcap \{X \setminus A : A \in \mathcal{A}\}$*;*
(ii) $X \setminus \cap \mathcal{A} = \bigcup \{X \setminus A : A \in \mathcal{A}\}$*.*

Corollary 2. *For any space X,*

(i) $\emptyset$ *and X are closed;*
(ii) *The union of finitely many closed sets is closed;*
(iii) *The intersection of any number of closed sets is closed.*

Definition. If X is a space, and $A \subset X$, then the *closure of* A, denoted by $\overline{A}$, is defined to be the intersection of all closed sets which contain A. Note that therefore $\overline{A}$ is always a closed set and A is closed iff $\overline{A} = A$.

If $x \in X$ and if every open set containing x contains a point of A other than x, we say that x *is a limit point of* A.

Theorem 3. *For any subset A of a space X, we have* $\overline{A} = A \cup \{x \in X : x \text{ is a limit point of } A\}$*.*

Theorem 4. *If A and B are subsets of a space X, then*

(i) $\overline{\emptyset} = \emptyset$*;*
(ii) $A \subset \overline{A}$*;*
(iii) $\overline{\overline{A}} = \overline{A}$*;*
(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$*.*

Theorem 5. *If $\{A_\alpha : \alpha \in \Lambda\}$ is any collection of subsets of a space X, then*

(i) $\overline{\bigcap_{\alpha \in \Lambda} A_\alpha} \subseteq \bigcap_{\alpha \in \Lambda} \overline{A_\alpha}$*;*
(ii) $\overline{\bigcup_{\alpha \in \Lambda} A_\alpha} \supseteq \bigcup_{\alpha \in \Lambda} \overline{A_\alpha}$*.*

Definition. If A is a subset of a space X, let $A^o = \{x \in A : x \in U \subset A \text{ for some open set } U\}$. We call A^o the *interior of* A, and if $x \in A^o$ we say that A

is a *neighborhood of* x. A point x is in the *boundary of* A if every open set containing x (equivalently, every neighborhood of x) meets both A and $X \setminus A$. We denote the set of boundary points of A by ∂A.

Theorem 6. *For any subset* A *of a space* X, $\overline{A} = A \cup \partial A = A^o \cup \partial A$.

Definition. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a subcollection of $\mathcal{T}$. We say $\mathcal{B}$ is a *base* for the topological space $(X, \mathcal{T})$ iff every member of $\mathcal{T}$ is a union of members of $\mathcal{B}$.

Theorem 7. *Let* X *be a set and let* $\mathcal{B}$ *be a collection of subsets of* X. *Then there is a (unique) topology* $\mathcal{T}$ *on* X *such that* $\mathcal{B}$ *is a base for* $(X, \mathcal{T})$ *if and only if*

(a) *Each* $x \in X$ *is in some member of* $\mathcal{B}$ *(i.e.,* $\mathcal{B}$ *covers* X*);*
(b) *Whenever* $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 \cap B_2$, *there exists* $B \in \mathcal{B}$ *with* $x \in B \subset B_1 \cap B_2$.

Definition. Collections $\mathcal{B}_1$ and $\mathcal{B}_2$ of subsets of a set X are *equivalent bases* iff there is a topology $\mathcal{T}$ on X such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are both bases for $(X, \mathcal{T})$.

Theorem 8. *Let* $(X, \mathcal{T})$ *be a topological space, let* $\mathcal{B}_1$ *be a base for* $(X, \mathcal{T})$, *and let* $\mathcal{B}_2$ *be a collection of subsets of* X. *If*

(i) $x \in B_1 \in \mathcal{B}_1 \Rightarrow \exists B_2 \in \mathcal{B}_2$ *such that* $x \in B_2 \subset B_1$, *and*
(ii) $x \in B_2 \in \mathcal{B}_2 \Rightarrow \exists B_1 \in \mathcal{B}_1$ *such that* $x \in B_1 \subset B_2$,

then $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are equivalent bases.*

Definition. A *subspace* of a topological space $(X, \mathcal{T})$ is a pair $(A, \mathcal{T}_A)$, where $A \subset X$ and $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$. $\mathcal{T}_A$ is called the *relative topology on* A, or the *topology on* A *induced by* $\mathcal{T}$.

Theorem 9. *If* $(X, \mathcal{T})$ *is a topological space,* $\mathcal{B}$ *a base for* $\mathcal{T}$, *and* $A \subset X$, *then the collection* $\{B \cap A : B \in \mathcal{B}\}$ *is a base for* $(A, \mathcal{T}_A)$.

Definition. Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a *metric* on X if for each $x, y, z \in X$, d satisfies:

(i) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$;
(ii) $d(x, y) = d(y, x)$ (i.e., d is *symmetric*);
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (the *triangle inequality*).

A *metric space* is a pair (X, d), where X is a set and d is a metric on X.

If $\epsilon > 0$, let $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ (if it is understood what metric we are talking about, we may omit the subscript d and write $B(x, \epsilon)$). $B_d(x, \epsilon)$ is the *ϵ-ball about x (w.r.t. d).*

Lemma 10. *Let* (X, d) *be a metric space. Then the collection* $\mathcal{B} = \{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$ *of* ϵ*-balls is a base for a topology on* X.

Lemma 10.5. *Let* (X, d) *be a metric space. Then a subset* U *of* X *is open in the metric topology iff, for every* $x \in U$, *there is* $\epsilon > 0$ *such that* $B_d(x, \epsilon) \subset U$.

Definition. Let (X, d) be a metric space. Then the topology on X given in Lemma 10 is called the *metric topology (generated by the metric d)*. If d and d' are two metrics on the same set X which generate the same topology, then d and d' are called *equivalent metrics*. A topological space $(X, \mathcal{T})$ is said to be *metrizable* if there is a metric on X which generates $\mathcal{T}$.

Theorem 11. *Let (X, d) be a metric space, and let $Y \subset X$. Let d_Y be the metric d restricted to $Y \times Y$. Then (Y, d_Y) is a metric space, and the metric topology on Y is the same as the subspace topology with respect to the metric topology on X.*

Definition. If $f : X \to Y$, where X and Y are spaces, and $x_0 \in X$, we say f *is continuous at* x_0 if, given any open set V in Y with $f(x_0) \in V$, there is an open set U in X with $x_0 \in U$ and $f(U) \subset V$. Also, f is *continuous* if f is continuous at every $x_0 \in X$.

Theorem 12. *If $f : X \to Y$, where X and Y are spaces, then the following are equivalent:*

(i) *f is continuous;*
(ii) *$f^{-1}(V)$ is open in X whenever V is open in Y;*
(iii) *$f^{-1}(C)$ is closed in X whenever C is closed in Y;*
(iv) *Whenever $A \subset X$ and $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$.*

Theorem 13. *If $f : X \to Y$ is continuous, and $A \subset X$, then the restriction of f to A, denoted $f \restriction A$, is a continous function from A (with the subspace topology) to Y.*

Theorem 14. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is $g \circ f : X \to Z$.*

Definition. A function $f : X \to Y$ is *closed* (resp., *open*) if the image of every closed set (resp., open set) in X is closed (resp., open) in Y.

Definition. A function $h : X \to Y$ is a *homeomorphism* if h is one-to-one, onto, and both h and h^{-1} are continuous. If such a homeomorphism exists, then the spaces X and Y are said to be *homeomorphic*.

Theorem 15. *If $h : X \to Y$ is one-to-one and onto, then the following are equivalent:*

(i) *h is a homeomorphism;*
(ii) *h is open and continuous;*
(iii) *h is closed and continuous.*

Exercise. Find an example of spaces X and Y, and a mapping $f : X \to Y$ which is one-to-one, onto, and continuous, but not a homeomorphism.

Definition. A *local base* at a point x of a space X is a collection $\mathcal{B}_x$ of open neighborhoods of x such that, whenever $x \in U$ where U is open, there is some $B \in \mathcal{B}_x$ with $x \in B \subset U$. A space X is *first-countable* if every point of x has a countable local base.

Remark. If $U_1, U_2, \dots$ is a countable local base at x, and we put $V_1 = U_1, V_2 = U_1 \cap U_2, V_3 = U_1 \cap U_2 \cap U_3, \dots$, then $V_1, V_2, \dots$ is another countable local base at x which is decreasing (in the sense that $V_1 \supset V_2 \supset V_3 \supset \dots$). So a point has a countable local base iff it has a countable decreasing local base.

Theorem 16. *Every metric space is first-countable.*

Definition. Let X be a topological space, $x \in X$, and $(x_n)_{n=1}^{\infty}$ a sequence of points in X. We say $(x_n)_{n=1}^{\infty}$ converges to x, and write $(x_n)_{n=1}^{\infty} \to x$, if every (open) neighborhood of x contains x_n for sufficiently large $n \in \mathbb{N}$ (i.e., for any (open) neighborhood N_x of x, there is $k \in \mathbb{N}$ such that $x_n \in N_x$ for all $n \geq k$).

Theorem 17. *Let X be a first-countable space. A point x is in the closure of a subset A of X iff there are $a_n \in A$, $n = 1, 2, \dots$, such that $(a_n)_{n=1}^{\infty} \to x$.*

Theorem 18. *Let X and Y be topological spaces, and $f : X \to Y$ a function.*

(a) *If f is continuous and $(x_n)_{n=1}^{\infty} \to x$ in X, then $(f(x_n))_{n=1}^{\infty} \to f(x)$ in Y.*
(b) *If X is first-countable, then f is continuous iff $(f(x_n))_{n=1}^{\infty} \to f(x)$ in Y whenever $(x_n)_{n=1}^{\infty} \to x$ in X.*

Definition. A space X is a T_0*-space* if whenever $x, y \in X$ with $x \neq y$, there is an open set containing one of these points but not the other. If there is always an open set containing x and missing y, then X is a T_1*-space*. If there are always disjoint open sets containing x and y, then X is a T_2*-space* or *Hausdorff space*.

A space X is a T_3*-space*, or *regular space*, if X is a T_1-space, and whenever $x \in X$ and H is a closed set not containing x, then there are disjoint open sets containing x and H, resp.

A space X is a $T_{3\frac{1}{2}}$*-space*, or *completely regular space*, if X is a T_1-space, and whenever $x \in X$ and H is a closed set not containing x, there is a continuous function $f : X \to [0, 1]$ with $f(x) = 0$ and $f(y) = 1$ for all $y \in H$.

X is a T_4*-space*, or *normal space*, if X is a T_1-space, and any two disjoint closed sets are contained in disjoint open sets.

Remark. It is easy to show and useful to note that a space X is regular iff, whenever $x \in X$ and H is a closed subset of X not containing x, there exists an open set U containing x whose closure misses H. Similarly, a T_1-space X is normal iff X is T_1 and, whenever H and K are disjoint closed subsets of X, there exists an open set U such that $U \supset H$ and $\overline{U} \cap K = \emptyset$.

Theorem 19. *$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$ and $T_{3\frac{1}{2}} \Rightarrow T_3$.*

Theorem 20. *A space X is a T_1-space iff every point of X is a closed set.*

Theorem 20.5. *Metrizable spaces are normal.*

Lemma 21. *Suppose H and K are disjoint closed subsets of a normal space X. Then one can assign to each rational number r with $0 < r < 1$ an open set U_r such that*

(i) *$H \subset U_r$ and $\overline{U_r} \cap K = \emptyset$;*
(ii) *$r < r' \Rightarrow \overline{U_r} \subset U_{r'}$.*

Theorem 22. *The following are equivalent for a T_1-space X:*

(a) *X is normal;*
(b) *Whenever H and K are disjoint closed subsets of X, there exists a continuous function $f : X \to [0, 1]$ with $f(x) = 0$ for all $x \in H$ and $f(x) = 1$ for all $x \in K$.*

Corollary 23. *$T_4 \Rightarrow T_{3\frac{1}{2}}$, i.e., every normal space is completely regular.*

Definition. Two subsets A and B of a space X are said to be *separated* if $\overline{A} \cap B = \emptyset = \overline{B} \cap A$ (i.e., no point of any one of the sets is in the closure of the other set). A T_1-space X is said to be *completely normal*, or a T_5*-space* if, given any two separated sets A and B, there are disjoint open sets containing A and B, respectively.

Theorem 24. *A space X is completely normal if and only if every subspace of X is normal.*

Remark. Sometimes "completely normal" is called "hereditarily normal"; of course this term is justified by Theorem 24.

Definition. A subset D of a topological space X is *dense* in X if $X = \overline{D}$ (equivalently, every non-empty open set contains a point of D). X is *separable* if it has a countable dense subset.

Definition. A space X is *second-countable* if it has a countable base.

Definition. A collection $\mathcal{U}$ of subsets of a space X is said to be a *cover* of X if every point of X is in some member of $\mathcal{U}$ (i.e., $X = \cup\mathcal{U}$). If $\mathcal{U}$ is a cover of X, then a subcollection $\mathcal{V}$ of $\mathcal{U}$ is a *subcover* if $\mathcal{V}$ is also a cover. A cover $\mathcal{U}$ is an *open cover* if every member of $\mathcal{U}$ is open.

Definition. A space X is *compact* (resp., *Lindelöf*) if every open cover of X has a finite (resp., countable) subcover.

Terminolgy. If P is a property of topological spaces, we say that a space X is *hereditarily P* iff every subspace of X has property P.

Theorem 25.

(a) *Every subspace of a second-countable space is second-countable;*
(b) *Every second-countable space is hereditarily separable and hereditarily Lindelöf.*

Theorem 26. *The following are equivalent for a metric space X:*

(a) *X is second-countable;*
(b) *X is separable;*
(c) *X is Lindelöf;*
(d) *X is hereditarily separable;*
(e) *X is hereditarily Lindelöf.*

Definition. Let A be a subset of a space X and let $x \in X$. If every neighborhood of x contains uncountably many points of A, then we call x a *complete accumulation point* of A.

Theorem 27.

(a) *Every infinite subset of a compact space X has a limit point in X;*
(b) *Every uncountable subset A of a Lindelöf space X has a complete accumulation point in X.*

Corollary 28. *For every uncountable subset A of a separable metric space, there is a complete accumulation point a of A such that $a \in A$.*

Definition. X is *countably compact* if every countable open cover of X has a finite subcover.

Theorem 29.. *The following are equivalent for a T_1-space X:*

(a) *X is countably compact;*
(b) *Every infinite subset of X has a limit point in X.*

Theorem 30. *A space X is compact iff X is both countably compact and Lindelöf.*

Definition. A space X is *sequentially compact* if every infinite sequence in X has an infinite convergent subsequence.

Theorem 31.

(a) *Sequentially compact spaces are countably compact.*
(b) *If X is first-countable, then X is countably compact iff X is sequentially compact.*

Theorem 32.. *The following are equivalent for a separable metric space X*[1] *:*

(a) *X is compact;*
(b) *X is countably compact;*
(c) *X is sequentially compact.*

Theorem 33.

(a) *Every continuous image of a compact (resp., Lindelöf) space is compact (resp., Lindelöf);*
(b) *Every closed subset of a compact (resp., Lindelöf) space is compact (resp., Lindelöf).*

Theorem 34. *Every compact subspace of a Hausdorff space is closed.*

Theorem 35. *Suppose $f : X \to Y$ is continuous, one-to-one, and onto. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Theorem 36. *Suppose $f : X \to \mathbb{R}$ is continuous, where X is a compact space. Then there exists $x_0 \in X$ such that $f(x_0) \geq f(x)$ for every $x \in X$.*

Theorem 37.

(a) *Every compact Hausdorff space is regular;*
(b) *Every compact Hausdorff space is normal.*

Definition. A space X is *locally compact* if every point has a compact neighborhood. (Equivalently, for each $x \in X$, there is a compact subset N of X such that $x \in N^o$.)

Theorem 37.5. *Let $(X, \mathcal{T})$ be a locally compact non-compact Hausdorff space. Then there is a compact Hausdorff space $(X^*, \mathcal{T}^*)$ satisfying:*

(i) *$X^* = X \cup \{\infty\}$, where $\infty \notin X$;*
(ii) *$\mathcal{T}^*_X = \mathcal{T}$ (i.e., the relative topology on X as a subspace of X^* is the same as $\mathcal{T}$);*
(iii) *U is an open nbhd of ∞ in X^* iff $X^* \setminus U$ is a compact subset of X.*

Remark. The space $(X^*, \mathcal{T}^*)$ in Theorem 37.5 is called the *one-point compactification* of the locally compact Hausdorff space $(X, \mathcal{T})$.

Theorem 38. *Every locally compact Hausdorff space is completely regular.*

Lemma 39. *Suppose A and B are subsets of a space X, and that there are open sets $U_0, U_1, U_2, \ldots$ and $V_0, V_1, V_2, \ldots$ such that:*

(i) *$A \subset \bigcup_{n \in \mathbb{N}} U_n$ and $B \subset \bigcup_{n \in \mathbb{N}} V_n$;*
(ii) *For each $n \in \mathbb{N}$, $\overline{U_n} \cap B = \emptyset = \overline{V_n} \cap A$.*

Then there are disjoint open sets U and V containing A and B, respectively.

[1] This result holds for all metric spaces, separable or not, but we are not in good position to prove this just now.

Theorem 40. *Every regular Lindelöf space is normal.*

Definition. X is said to be *perfectly normal* if X is normal and every closed set H in X is a G_δ-set (i.e., $H = \bigcap_{n\in\mathbb{N}} U_n$ for some sequence $U_1, U_2, U_3, \dots$ of open sets).

Theorem 41. *Metrizable spaces are perfectly normal.*

Theorem 42. *The following are equivalent for a T_1-space X:*

(a) *X is perfectly normal;*
(b) *For any closed subset H of X, there is a sequence $U_0, U_1, U_2, \dots$ of open sets such that $H = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \overline{U_n}$;*
(c) *For any closed subset H of X, there is a continuous function $f : X \to [0,1]$ such that $H = f^{-1}(0)$;*
(d) *Whenever H and K are disjoint closed subsets of X, there exists a continuous function $f : X \to [0,1]$ with $H = f^{-1}(0)$ and $K = f^{-1}(1)$.*

Theorem 43. *Every subspace of a perfectly normal space is perfectly normal, and hence perfectly normal spaces are completely normal.*

Lemma 44. *Suppose we have a decreasing sequence $H_1 \supset H_2 \supset \dots$ of non-empty closed sets, and H_1 is compact. Then $\bigcap_{n\in\mathbb{N}} H_n \neq \emptyset$.*

Theorem 45. *Let X be a locally compact Hausdorff space. If $G_1, G_2, \dots$ are dense open subsets of X, then $\bigcap_{n\in\mathbb{N}} G_n$ is dense.*

Remark. A space with the property of Theorem 45 is called a *Baire space.* So all locally compact Hausdorff spaces, in particular, $\mathbb{R}^n$ for all n, are Baire spaces.

Corollary 46. *Suppose X is a locally compact Hausdorff space (or, just a Baire space), and $X = \bigcup_{n\in\mathbb{N}} X_n$. Then for some $n \in \mathbb{N}$, $\overline{X_n}^o \neq \emptyset$ (i.e., some X_n is dense is some open subset of X).*

Definition. Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a collection of spaces. Let $X = \Pi_{\alpha\in\Lambda} X_\alpha$. (So, a point of X can be thought of as a sequence $(x_\alpha)_{\alpha\in\Lambda}$, where $x_\alpha \in X_\alpha$ for all α, or equivalently as a function $f : \Lambda \to \cup_{\alpha\in\Lambda} X_\alpha$ with $f(\alpha) \in X_\alpha$ for all α.)

Let $\mathcal{B}$ be the collection of all sets of the form $\Pi_{\alpha\in\Lambda} U_\alpha$, where $U_\alpha \in \mathcal{T}_\alpha$ for all α, and $U_\alpha = X_\alpha$ for all but finitely many α. Then $\mathcal{B}$ is a base for a topology on X; this topology is called the *product topology* and X with this topology is called the *product space* of $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$. Each X_α is called a *coordinate space* of X.

Observation: If for each α, $\mathcal{B}_\alpha$ is a base for X_α, then the collection of all sets of the form $\Pi_{\alpha\in\Lambda} B_\alpha$, where $B_\alpha = X_\alpha$ for all but finitely many α and $B_\alpha \in \mathcal{B}_\alpha$ if $B_\alpha \neq X_\alpha$, is an equivalent base for the product topology.

Definition. Let $\Pi_{\alpha\in\Lambda} X_\alpha$ be a product space. For each $\alpha \in \Lambda$, the function $\pi_\alpha : \Pi_{\alpha\in\Lambda} X_\alpha \to X_\alpha$ defined by $\pi_\alpha((x_\beta)_{\beta\in\Lambda}) = x_\alpha$ is called the *projection map onto* X_α.

Note that if $U_\alpha \subset X_\alpha$, then $\pi_a^{-1}(U_\alpha) = \Pi_{\beta\in\Lambda} V_\beta$, where $V_\alpha = U_\alpha$, and $V_\beta = X_\beta$ if $\beta \neq \alpha$. Thus a typical basic open set in the product topology can be denoted by $\bigcap_{i\leq n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$, where $\alpha_0, \alpha_1, \dots, \alpha_n$ are in Λ and U_{α_i} is open in X_{α_i} for each $i \leq n$.

Theorem 47. *Projection maps are always continuous and open.*

Theorem 48. *Let $X = \Pi_{\alpha\in\Lambda} X_\alpha$ be a product space, and let $\vec{x} = (x_\alpha)_{\alpha\in\Lambda}$ be a point in X. Fix $\alpha_0 \in \Lambda$. Then the "cross-section"*

$$\{\vec{y} \in X : \forall\alpha \neq \alpha_0 (y_\alpha = x_\alpha)\}$$

is homeomorphic to X_{α_0}.

Theorem 49. *Let $X = \Pi_{\alpha\in\Lambda} X_\alpha$ be a product space, and let $A_\alpha \subset X_\alpha$ for all α. Then*

$$\overline{\Pi_{\alpha\in\Lambda} A_\alpha} = \Pi_{\alpha\in\Lambda}\overline{A_\alpha}.$$

Theorem 50. *Let $X = \Pi_{\alpha\in\Lambda} X_\alpha$ be a product space. If each X_α is a T_i-space, where $i = 0, 1, 2, 3$ or 3.5, then so is X.*

Example 51. *The Sorgenfrey line S is a regular Lindelöf space (hence normal[2]), but S^2 is neither Lindelöf nor normal.*

Lemma 52(Tube Lemma). . *Let X be a space, and let Y be compact. Fix $x_0 \in X$. Then for any collection $\mathcal{U}$ of open sets in $X \times Y$ such that $\{x_0\} \times Y \subset \cup\mathcal{U}$, there is some finite subcollection $\mathcal{V}$ of $\mathcal{U}$ and an open nbhd O of x_0 such that*

$$O \times Y \subset \cup\mathcal{V}.$$

Theorem 53. *If X and Y are compact, so is $X \times Y$.*

Remark. In fact it's true that any product of compact spaces is compact. But the argument, which we'll do later, is significantly more complicated than that of Theorem 53.

Theorem 54. *If X_n is a first-countable (resp., second-countable) space for each $n \in \mathbb{N}$, then $\Pi_{n\in\mathbb{N}} X_n$ is first-countable (resp., second-countable).*

Theorem 55. *If X_n is separable for each $n \in \mathbb{N}$, then $\Pi_{n\in\mathbb{N}} X_n$ is separable.*[3]

Lemma 56. *Let (X, d) be a metric space and let $M > 0$. Define d' by $d'(x, y) = M$ if $d(x, y) \geq M$ and $d'(x, y) = d(x, y)$ otherwise. Then d' is an equivalent metric on X.*

Theorem 57. *Let $(X_1, d_1), (X_2, d_2), \ldots$ be metric spaces such that, for each $n \in \mathbb{N}$, $d_n(x, y) \leq 1/2^n$ for every $x, y \in X_n$. For $\vec{x}, \vec{y} \in \Pi_{n\in\mathbb{N}} X_n$, define*

$$d(\vec{x}, \vec{y}) = max_{n\in\mathbb{N}} d_n(x_n, y_n).$$

Then d is a metric on $\Pi_{n\in\mathbb{N}} X_n$ and the topology generated by d is the same as the product topology.

Corollary 58. *If X_n is a metrizable space for each $n \in \mathbb{N}$, then $\Pi_{n\in\mathbb{N}} X_n$ is metrizable.*

[2]In fact, S is perfectly normal.

[3]In fact, if X_i is separable for every $i \in [0, 1]$, then $\Pi_{i\in[0,1]} X_i$ is separable.

Definition. A space X is *connected* iff X cannot be written as the union of two disjoint nonempty open sets (equivalently, X has no proper nonempty subset which is both open and closed).

Theorem 59. *Let $(X, \mathcal{T})$ be a topological space, and let $C \subset X$. Then the following are equivalent:*

(a) *$(C, \mathcal{T}_C)$ is connected;*
(b) *C is not the union of two nonempty separated sets.*

Remark. It follows from Theorem 59 (by taking $C = X$) that a space X is connected iff X is not the union of two nonempty separated sets. This is often taken as the definition of connected.

Theorem 60. *A subspace X of the real line is connected iff X is an interval (in the broad sense of the term, i.e., open, closed, or half-open, and also allowing infinite intervals).*

Definition. A space X *path-connected* iff for any two points a, b in X, there is a continuous function $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$. The function f is called a *path* from a to b.

Theorem 61. *The continuous image of a connected (resp., path-connected) space is connected (resp., path connected).*

Theorem 62. *Let X be a space, $A \subset X$, and suppose $A \subseteq B \subseteq \overline{A}$. If A is connected, so is B. In particular, the closure of a connected set is connected.*

Theorem 63. *Suppose $\{A_\alpha : \alpha \in \Lambda\}$ is a collection of connected (resp., path connected) subspaces of a space X, and for any $\alpha, \beta \in \Lambda$, $A_\alpha \cap A_\beta \neq \emptyset$. Then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected (resp., path connected).*

Theorem 64. *Every path connected space is connected.*

Example. The *topologist's sine curve* is the following subspace of the plane: $(\{0\} \times [-1,1]) \cup \{(x, sin\ 1/x) : 0 < x \leq 1\}$.

Theorem 65. *The topologist's sine curve is connected but not path connected.*

Theorem 66. *Let $X = \Pi_{\alpha \in \Lambda} X_\alpha$ be a product space, and let Y be any space. A function $f : Y \to X$ is continuous iff $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous for each $\alpha \in \Lambda$.*

Theorem 67. *Any product of path connected spaces is path connected.*

Theorem 68. *Any product of connected spaces is connected .*

Definition. Let X be a space, and $x \in X$. The component $C(x)$ of x is the union of all connected subsets of X which contain x. Similarly, the path component $P(x)$ of x is the union of all path connected subsets of X which contain x.

Theorem 69.

(a) *The component (resp., path component) of a point x is the largest connected (resp., path connected) subset of X which contains x;*
(b) *The collection of components (resp., path components) of a space X forms a partition of X (i.e., they are pairwise disjoint and their union is X) into connected (resp., path connected) subsets of X ;*
(c) *Each component of X is closed in X.*

Remark. Unlike components, path components need not be closed: it is easy to see that the topologist's sine curve has a path component which is not closed.

Definition. A space X is *locally connected* (resp., *locally path connected*) if whenever $x \in U$ where U is open in X, there is a connected (resp., path connected) subset N of U such that $x \in N^o$.

Remark. The topologist's sine curve is an example of a connected space which is not locally connected.

Lemma 70. *The components (resp., path components) of a locally connected (resp., locally path connected) space are open.*

Theorem 71. *A space X is locally connnected (resp., locally path connected) iff X has a base consisting of connected (resp., path connected) subsets.*

Theorem 72. *If X is locally path connected, then the components and path components of X are the same. In particular, a connected space which is locally path connected is path connected.*

Definition. Let X be a space and $x \in X$. The *quasicomponent* $Q(x)$ of x is the intersection of all closed-and-open (or "clopen") sets containing x.

Theorem 73. *For any space X and $x \in X$, the component $C(x)$ of x is contained in the quasicomponent $Q(x)$ of x.*

Lemma 74. *Let X be a compact space, let $\mathcal{C}$ be a collection of closed subsets of X, and let $H = \bigcap \mathcal{C}$. If $H \subset O$, where O is open, then there is a finite subcollection $\mathcal{C}'$ of $\mathcal{C}$ such that $H \subseteq \bigcap \mathcal{C}' \subseteq O$.*

Theorem 75. *If X is a compact Hausdorff space, then the components of X are the same as the quasicomponents of X.*

Remark. Let X be the following subspace of the plane: $\{(0,0),(0,1)\} \cup \bigcup_{n=1}^{\infty}(\{1/n\} \times [0,1])$. Then the component in X of the point $(0,0)$ is itself, but the quasicomponent of $(0,0)$ is $\{(0,0),(0,1)\}$.

Theorem 76. *If $K_1, K_2, K_3, \ldots$ are compact, connected subsets of a Hausdorff space X, and $K_1 \supseteq K_2 \supseteq \ldots$, then $K = \bigcap_{n=1}^{\infty} K_n$ is compact and connected.*

Remark. A compact connected metrizable space is called a *continuum*. A compact connected Hausdorff space is sometimes called a *Hausdorff continuum* (in this case, the adjective "Hausdorff" emphasizes that the space may be nonmetrizable).

Theorem 77. *Let H be a proper closed subspace of a compact connected Hausdorff space X. Then any component of the subspace H meets the boundary of H.*

Definition. A collection $\mathcal{F}$ of subsets of a set X is said to have the *finite intersection property (f.i.p.)* if $\cap\mathcal{F}' \neq \emptyset$ for any finite subcollection $\mathcal{F}'$ of $\mathcal{F}$.

Theorem 78. *A space X is compact if and only if for every collection $\mathcal{F}$ of closed sets with the f.i.p., $\cap\mathcal{F} \neq \emptyset$.*

Definition. A collection $\mathcal{F}$ of subsets of a set X is called a *filter on X* if:

(i) Whenever $F_1, F_2, \ldots, F_n \in \mathcal{F}$, then $\bigcap_{i=1}^{n} F_i \in \mathcal{F}$;
(ii) If $F \in \mathcal{F}$ and $F \subset G \subset X$, then $G \in \mathcal{F}$;
(iii) $\emptyset \notin \mathcal{F}$.

In other words, a filter is a collection of nonempty sets which is closed under supersets and finite intersections.

Also, a filter $\mathcal{F}$ on X is called an *ultrafilter* if $\mathcal{F}$ is not properly contained in any other filter on X.

Theorem 79. *Let $\mathcal{F}$ be a filter on X. Then the following are equivalent:*

(a) *$\mathcal{F}$ is an ultrafilter;*
(b) *If $G \subset X$ and $G \cap F \neq \emptyset$ for every $F \in \mathcal{F}$, then $G \in \mathcal{F}$;*
(c) *For every $G \subset X$, either $G \in \mathcal{F}$ or $X \setminus G \in \mathcal{F}$.*

Theorem 80. *Suppose $\mathcal{F}$ is a collection of subsets of X having the f.i.p. Then $\mathcal{F}$ is contained in some maximal collection $\mathcal{G}$ of subsets of X with the f.i.p.; furthermore, any such $\mathcal{G}$ is an ultrafilter on X.*

Definition. Let X be a space, $p \in X$, and $\mathcal{F}$ a filter on X. We say that $\mathcal{F}$ *clusters at p* if $p \in \overline{F}$ for every $F \in \mathcal{F}$, and we say $\mathcal{F}$ *converges to p* if every nbhd of p contains a member of $\mathcal{F}$ (equivalently, every nbhd of p is in $\mathcal{F}$).

Theorem 81. *The following are equivalent for a space X:*

(a) *X is compact;*
(b) *Every filter on X clusters at some point;*
(c) *Every ultrafilter on X converges to some point.*

Theorem 82 (Tychonoff Theorem). *Any product of compact spaces is compact.*

Definition. Let $\mathcal{F}$ be a family of continuous functions from a space X to the unit interval $I = [0, 1]$. We say $\mathcal{F}$ *separates points* if, given $x_1 \neq x_2 \in X$, there exists $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$. We say $\mathcal{F}$ *separates points from closed sets* if, given $x \in X$ and any closed set H with $x \notin H$, there exists $f \in \mathcal{F}$ with $f(x) \notin \overline{f(H)}$. The *evaluation function determined by* $\mathcal{F}$ is the function $e_{\mathcal{F}} : X \to I^{\mathcal{F}}$ defined by $e_{\mathcal{F}}(x) =< f(x) >_{f \in \mathcal{F}}$.

Theorem 83. *Let $\mathcal{F}$ be a collection of continuous functions from X into the unit interval I. Then:*

(a) *$e_{\mathcal{F}}$ is continuous;*
(b) *If $\mathcal{F}$ separates points, then $e_{\mathcal{F}}$ is one-to one;*
(c) *If X is a T_1-space and $\mathcal{F}$ separates points from closed sets, then $e_{\mathcal{F}} : X \to e_{\mathcal{F}}(X)$ is a homeomorphic embedding of X into $I^{\mathcal{F}}$.*

Theorem 84. *The following are equivalent for a space X:*

(a) *X is a separable metrizable space;*
(b) *X is a regular 2^{nd}-countable space;*
(c) *X is homeomorphic to a subspace of the Hilbert cube $I^{\mathbb{N}}$.*

Theorem 85. *A space X is completely regular if and only if X is homeomorphic to a subspace of I^{κ} for some cardinal κ.*

Definition. A collection $\mathcal{U}$ of subsets of a space X is said to be *point-finite* (resp., *locally finite*) if, for each $x \in X$, the set $\{U \in \mathcal{U} : x \in U\}$ is finite (resp., there is a nbhd O_x of x such that the set $\{U \in \mathcal{U} : O_x \cap U \neq \emptyset\}$ is finite). A collection $\mathcal{U}$ of subsets of X is called *closure-preserving* if, for any $\mathcal{U}' \subset \mathcal{U}$, we have

$$\cup\{\overline{U} : U \in \mathcal{U}'\} = \overline{\cup\{U : U \in \mathcal{U}'\}}.$$

Lemma 86. *Suppose $\mathcal{U}$ is a locally finite collection of subsets of X. Then:*

(a) *$\mathcal{U}$ is closure-preserving;*
(b) *If X is compact, then $\mathcal{U}$ is finite.*

Exercise (a) Find a locally finite cover of the real line $\mathbb{R}$ by open intervals;
(b) Find a point-finite cover of $\mathbb{R}$ which is not locally finite.
(c) Find a point-finite open cover of $[0, 1]$ which is not closure-preserving.
(d)Find an infinite collection of subsets of $[0, 1]$ which is closure-preserving.

Definition. Let $\mathcal{U}$ be a cover of a space X. A cover $\mathcal{V}$ of X is called a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$, there is some $U \in \mathcal{U}$ with $V \subseteq U$.

Definition. A space X is said to be *paracompact* (resp., *metacompact*) if every open cover $\mathcal{U}$ of X has a locally finite (resp., point-finite) open refinement $\mathcal{V}$.

Theorem 87.

(a) *Every paracompact T_2-space is regular;*
(b) *Every paracompact T_2-space is normal.*

Theorem 88. *Every regular Lindelöf space is paracompact.*

Theorem 88.5. *The following are equivalent for a regular space X:*

(a) *X is paracompact;*
(b) *Every open cover of X has a locally finite refinement;*
(c) *Every open cover of X has a locally finite closed refinement.*

Theorem 89. *Let X be a regular space. If every open cover of X has an open refinement $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$, where each $\mathcal{V}_n$ is locally finite, then X is paracompact.*

Remark. A collection that is the union of countably many locally finite collections is usually called a *σ-locally finite* collection. However, Munkres calls it a *countably locally finite* collection.

Lemma 90. *Let X be metrizable with metric d. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an open cover of X indexed by the ordinal κ. For each $\alpha < \kappa$ and $n \in \mathbb{N}$, let*

$$H_{\alpha,n} = \{x \in U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta : d(x, X \setminus U_\alpha) \geq 1/n\},$$

and let

$$V_{\alpha,n} = \bigcup_{x \in H_{\alpha,n}} B_d(x, 1/4n).$$

Then for each n, $\mathcal{V}_n = \{V_{\alpha,n} : \alpha < \kappa\}$ is locally finite (in fact, every point of X has a nbhd meeting at most one member of $\mathcal{V}_n$), and $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a refinement of $\mathcal{U}$.

Corollary 91. *Every metrizable space is paracompact.*

Remark. Note the use of the Axiom of Choice (in the form of the well-ordering principle) in the proof of Corollary 91.

Theorem 92. *Let $I = [0, 1]$ and let Λ be an index set. Define a metric on I^Λ by*

$$d(\vec{x}, \vec{y}) = sup\{|x_\alpha - y_\alpha| : \alpha \in \Lambda\}.$$

Then the metric topology on I^Λ is finer than the usual product topology on I^Λ (i.e., every set open in the product topology is also open in the metric topology).

Remark. The metric on I^Λ defined in Theorem 92 is sometimes called the *uniform metric*. (E.g., Munkres calls it that.)

Lemma 93. *If X is regular and has a σ-locally finite basis $\mathcal{B}$ (i.e., $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$, where each $\mathcal{B}_n$ is locally finite), then X is perfectly normal.*

Lemma 94. *Suppose X is regular and has a basis $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$,where each $\mathcal{B}_n$ is locally finite. For each n and each $B \in \mathcal{B}_n$, let $f_{n,B} : X \to [0, 1/n]$ be continuous such that $f_{n,B}(x) = 0 \iff x \notin B$, and let*

$$\mathcal{F} = \{f_{n,B} : n \in \mathbb{N}, B \in \mathcal{B}_n\}.$$

Then the evaluation map $e_\mathcal{F} : X \to I^\mathcal{F}$ is a homeomorphic imbedding of X into $I^\mathcal{F}$, where $I^\mathcal{F}$ is given the uniform metric topology.

Theorem 95. *A space X is metrizable iff X is regular and has a σ-locally finite basis.*

Theorem 96. *Let X be a T_1-space. Then X is compact iff X is paracompact and countably compact.*

Theorem 97. *The following are equivalent for a metrizable space X:*

(a) *X is compact;*
(b) *X is countably compact;*
(c) *X is sequentially compact.*

Definition. A space X is *locally metrizable* if every point of X has a metrizable neighborhood.

Theorem 98. *If a paracompact T_2-space is locally metrizable, then it is metrizable.*

Definition. Let X be a metric space with metric d. A sequence $(x_n)_{n=1}^{\infty}$ is *d-Cauchy* if for each $\epsilon > 0$, there is $k \in \omega$ such that $d(x_i, x_j) \leq \epsilon$ whenever $i, j \geq k$. The metric d is *complete* if every d-Cauchy sequence converges. A metrizable space X is *completely metrizable* if there is a complete metric d for X (i.e., d is complete and generates the given topology on X).

Remarks: The usual metric on $\mathbb{R}^n$ is complete. The usual metric on the space of irrationals is not complete, but the irrationals are completely metrizable (this fact will follow from subsequent results).

Theorem 99. *TFAE for a metric d:*

(a) *d is complete;*
(b) *Whenever $(A_n)_{n<\omega}$ is a decreasing sequence of closed sets with $diam(A_n) \to 0$, then $\bigcap_{n<\omega} A_n \neq \emptyset$;*

(The diameter $diam(A)$ of a subset A of a metric space (X, d) is defined by $diam(A) = sup\{d(x, y) : x, y \in A\}$.)

Corollary 100. *If X is compact and metrizable, then every metric which generates the topology is complete.*

Theorem 101. *A completely metrizable space is a Baire space.*

(Recall that X is a *Baire space* if the intersection of countably many dense open sets is always dense.)

Theorem 102. *If d is a complete metric for a space X and $A \subset X$, then $d \upharpoonright A \times A$ is a complete metric for the subspace A iff A is closed in X.*

Theorem 103. *A countable product of completely metrizable spaces is completely metrizable.*

Definition. Given a space X, define $C^*(X)$ to be the set of all bounded continuous real-valued functions on X. For $f, g \in X$, define $d(f, g) = sup_{x \in X}|f(x) - g(x)|$.

Theorem 104. *The function $d : C^*(X) \times C^*(X) \to \mathbb{R}$ defined above is a complete metric on $C^*(X)$.*

Remark. Theorem 104 doesn't work for $C(X)$, the set of all (not necessarily bounded) continuous real-valued functions on X, because in this case the "metric" d as defined for $C^*(X)$ wouldn't necessarily be defined for all $f, g \in C(X)$ (the sup could be infinite). However, note that it does work if X is compact, for then $C(X)$ and $C^*(X)$ are the same (if X is compact and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ is compact, hence bounded).

Remark. $C^*(X)$ is a vector space over $\mathbb{R}$ under the usual operations of addition of functions and scalar multiplication. Also, $C^*(X)$ has a "norm" $|| \; ||$ defined by

$$||f|| = sup_{x \in X}|f(x)|.$$

I.e., $|| \; ||$ satisfies:

(i) $||f|| \geq 0$ and $||f|| = 0$ iff $f = 0$;
(ii) $||f + g|| \leq ||f|| + ||g||$;
(iii) $||cf|| = |c| \cdot ||f||$ for all $c \in \mathbb{R}$.

So $C^*(X)$ is what is called a *normed linear space.* Given a norm, one can define a metric d by declaring $d(f, g) = ||f - g||$. For the norm $||f|| = sup_{x \in X}|f(x)|$ on $C^*(X)$, the associated metric is precisely the metric on $C^*(X)$ defined earlier. A *Banach space* is a complete normed linear space, i.e., complete in the metric defined by the norm. So by Theorem 104, $C^*(X)$ is a Banach space.

Lemma 105.

(a) *Let U be an open subset of a metrizable space X with metric d. Then the map defined by $x \to (x, 1/d(x, X \setminus U))$ is a homeomorphism of U onto a closed subset of $X \times \mathbb{R}$.*
(b) *Let G be a G_δ-subset of a metrizable space X. Then G is homeomorphic to a closed subset of $X \times \mathbb{R}^{\mathbb{N}}$.*

Corollary 106. *Any G_δ-subset of a completely metrizable space is completely metrizable.*

Remark. We won't do this (neither does Munkres), but the converse of Corollary 106 is also true; i.e., a subspace Y of a completely metrizable space X is completely metrizable if and only if Y is G_δ in X.

Definition. Let (X, d) and (X', d') be metric spaces. A mapping $j : X \to X'$ is called an *isometry* if for each $x, y \in X$, $d'(j(x), j(y)) = d(x, y)$ (i.e., j doesn't change distances).

Theorem 107. *Let (X, d) be a metrizable space and let $a \in X$. For each $x \in X$, define $f_x \in C^*(X)$ by*

$$f_x(z) = d(z, x) - d(z, a).$$

The the mapping $j : X \to C^*(X)$ defined by $j(x) = f_x$ is an isometry (under the usual distance on $C^*(X)$).

Remark. If d were a bounded metric, defining $f_x(z) = d(z, x)$ would do.

Corollary 108. *Every metric space (X, d) is isometrically embeddable in a complete metric space.*

Theorem 109. *For every metric space (X, d), there is a unique (up to isometry) complete metric space $(\tilde{X}, \tilde{d})$ such that $\tilde{X}$ contains X as a dense subspace and $\tilde{d} \restriction X \times X = d$.*

Note: It follows readily from Corollary 108 that such an $(\tilde{X}, \tilde{d})$ exists (consider the closure of the image of X under the isometric embedding). So you need to show uniqueness.

Definition and Remark. The metric space $(\tilde{X}, \tilde{d})$ given by Theorem 109 is called the *completion* of (X, d). It can also be defined (by a method analogous to a classical way of defining the reals from the rationals) by "adding a point" to X for each Cauchy sequence in X which does not converge in X, calling two Cauchy sequences $(x_n), (y_n)$ equivalent if $d(x_n, y_n) \to 0$. Each "point" of $\tilde{X} \setminus X$, then, is an equivalence class of Cauchy sequences in X. The distance $\tilde{d}$ between two equivalence classes with representatives (x_n) and (y_n) is defined by $\tilde{d}((x_n), (y_n)) = lim_{n\to\infty} d(x_n, y_n)$.

Lemma 110. *If a locally compact space Y is a dense subspace of a Hausdorff space X, then Y is open in X.*

Corollary 111. *Every locally compact metrizable space is completely metrizable.*

Definition. Let X and Y be spaces, and $q : X \to Y$ a surjective map. Then q is called a *quotient map* provided

$$U \text{ is open in } Y \text{ iff } q^{-1}(U) \text{ is open in } X.$$

Note that quotient maps are continuous.

Theorem 112. *Open continuous surjections and closed continuous surjections are quotient maps.*

Theorem 113. *Let $q : X \to Y$ be a quotient map, and $f : Y \to Z$ arbitrary. Then f is continuous iff $f \circ q : X \to Z$ is continous.*

Definition. Let $q : X \to Y$ be a quotient map. A subset U of X is called *saturated* if $U = q^{-1}(q(U))$ (equivalently, U meets a fiber $q^{-1}(y)$ iff U contains it).

Theorem 114. *Let $q : X \to Y$ be a quotient map. Then the collection of open sets of Y is equal to*

$$\{q(U) : U \text{ is a saturated open subset of } X\}.$$

Definition.

(a) Let X be a space, Y a set, and $f : X \to Y$ a surjection. Let

$$\mathcal{T}_f = \{U \subset Y : f^{-1}(U) \text{ is open in } X\}.$$

Then $\mathcal{T}_f$ is easily seen to be a topology on Y, under which f is a quotient map. $\mathcal{T}_f$ is called the *quotient topology induced by f*.

(b) Let X be a space, and let $\mathcal{D}$ be a partition of X (i.e., $\mathcal{D}$ is a pairwise-disjoint cover of X). Define $p : X \to \mathcal{D}$ by $p(x) = D \iff x \in D$. Then $\mathcal{D}$ with the quotient topology induced by p is called a *quotient space* (or *decomposition space*, or *indentification space*) of X. We say that $\mathcal{D}$ is the space obtained from X by identifying points $x, y \in X$ if they are in the same member of $\mathcal{D}$. If $\mathcal{D}$ has only one element A with more than one point, the resulting quotient space is also denoted by X/A.

(c) Let X be a space and let $\sim$ be an equivalence relation on X. Then $X/\sim$ denotes the quotient space of X where $\mathcal{D}$ is the collection of $\sim$-equivalence clases.

Exercise. (a) Let $X = [0, 2\pi]$, and let $A = \{0, 2\pi\}$. Show that X/A is homeomorphic to the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

(Similarly, the space obtained from the unit disk $B^2 = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ by identifying its boundary S^1 to a single point is homeomorphic to the 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.)

(b) Let $X = \mathbb{N} \times [0, 1]$ with its topology as a subspace of the plane. Note that each $\{n\} \times [0, 1]$ is open and closed in X (we say X is a *topological sum* of countably many copies of $[0, 1]$). Let $A = \mathbb{N} \times \{0\}$. Describe the topology of X/A (i.e., what do the neighborhoods of points of X/A look like?). Is X/A metrizable?

Example. Let $X = [0, 1] \times [0, 1]$.

(a) If we identify $(x, 0)$ with $(x, 1)$ for each $x \in [0, 1]$, the resulting quotient space is a cylinder.
(b) If we also identify $(0, y)$ with $(1, y)$ for each y, the resulting quotient space is a torus.
(c) If we identify $(x, 0)$ with $(1 - x, 1)$, the resulting quotient space is a *Möbius strip*.
(d) If we identify as in (c), and also identify $(0, y)$ with $(1, y)$, the result is a *Klein bottle*.

Definition of the Cantor "middle thirds" set C. For each finite sequence σ of 0's and 1's (including the empty sequence $\emptyset$), we define a closed subinterval I_σ of $[0, 1]$ as follows. Start by setting $I_\emptyset = [0, 1]$. Then if I_σ has been defined, let $I_{\sigma^\frown 0}$ and $I_{\sigma^\frown 1}$ be the left and right thirds, respectively, of I_σ. Thus $I_0 = [0, 1/3]$, $I_1 = [2/3, 1]$, $I_{00} = [0, 1/9]$, $I_{01} = [2/9, 1/3]$, etc. For each n, let $C_n = \cup\{I_\sigma : \sigma \text{ has length } n\}$. So, $C_0 = [0, 1]$, $C_1 = [0, 1/3] \cup [2/3, 1]$, etc. Finally, $C = \bigcap_{n \in \mathbb{N}} C_n$.

Theorem 115. *The Cantor set C as defined above is (as a subspace of the real line $\mathbb{R}$) compact, metrizable, has no isolated points, and has a countable base of open and closed sets. It is an uncountable closed subset of the $\mathbb{R}$ with empty interior in $\mathbb{R}$.*

Remark. A space which has a base of open and closed sets is sometimes called *zero-dimensional*, or more precisely, is said to have *small inductive dimension zero*. (There are several concepts of dimension.) A closed set–especially a closed subset of $\mathbb{R}$–with no isolated points is sometimes called *perfect*.

Theorem 116. *Suppose X is compact Hausdorff, has no isolated points, and has a countable base of open and closed sets. Then X is homeomorphic to the Cantor set.*

Corollary 117. *The following spaces are homeomorphic to the Cantor set C:*

(i) $\{0, 1\}^{\mathbb{N}}$;
(ii) $C \times C$;
(iii) $C^{\mathbb{N}}$.

Theorem 118. *The function $f : 2^{\mathbb{N}} \to [0, 1]$ defined by $f(\vec{x}) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$ is continuous and onto.*

Theorem 119(Space-filling curve). *There is a continuous function from $[0, 1]$ onto $[0, 1]^2$.*