A survey of D-spaces

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Abstract. A space X is a D-space if whenever one is given a neighborhood $N(x)$ of x for each $x \in X$, then there is a closed discrete subset D of X such that $\{N(x) : x \in D\}$ covers X. It is a decades-old open question whether some of the standard covering properties, such as Lindelöf or paracompact, imply D . This remains unsettled, yet there has been a considerable amount of interesting recent work on D-spaces. We give a survey of this work.

1. Introduction

A *neighborhood assignment* for a space (X, τ) is a function $N : X \to \tau$ with $x \in N(x)$ for every $x \in X$. X is said to be a *D-space* if for every neighborhood assignment N, one can find a closed discrete $D \subset X$ such that $N(D) = \{N(x) : x \in D\}$ covers X, i.e., $X = \bigcup_{x \in D} N(x) = \bigcup N(D)$.

The notion of a D-space seems to have had its origins in an exchange of letters between E.K. van Douwen and E. Michael in the mid-1970's,[1] but the first paper on D-spaces is a 1979 paper of van Douwen and W. Pfeffer [**25**]. The D property is a kind of covering property; it is easily seen that compact spaces and also σ-compact spaces are D-spaces, and that any countably compact D-space is compact.

Part of the fascination with D-spaces is that, aside from these easy facts, very little else is known about the relationship between the D property and many of the standard covering properties. For example, it is not known if a very strong covering property such as hereditarily Lindelöf implies D, and yet for all we know it could be that a very weak covering property such as submetacompact or submetalindelöf implies D!

While these questions about covering properties remain unsettled, there nevertheless has been quite a lot of interesting recent work on D-spaces. The purpose of this article is give a survey of this work. In Section 2, we describe some important examples of spaces which are not D, in Section 3 we discuss what is known (and not) about the relationship of covering properties to D-spaces, in Section 4 we list various results which say that spaces which have certain generalized metric properties or base properties are D-spaces, Sections 5, 6, and 7 are about unions,

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[1] Michael sent van Douwen a letter with a proof that semistratifiable spaces are D, and van Douwen replied with an alternate proof. Michael kindly provided me with a copy of van Douwen's letter to him, which is dated June 6, 1975, and another letter, with undetermined date, in which van Douwen proves that strong Σ-spaces are D. These results, however, did not appear in the literature until 1991 [**16**] and 2002 [**13**], respectively.

products, and mappings of D-spaces, respectively, and finally Section 8 is on some other notions closely related to D-spaces.

All spaces are assumed to be regular and T_1.

2. Examples

To help get a feeling for the property, we start by discussing ways to recognize spaces that aren't D and give a couple of important examples. We mentioned in the introduction that countably compact D-spaces must be compact. This illustrates a more general fact. Recall that the *extent*, $e(X)$ of a space X is the supremum of the cardinalities of its closed discrete subsets, and the *Lindelöf degree*, $L(X)$, is the least cardinal κ such that every open cover of X has a subcover of cardinality $\leq \kappa$. Note that $e(X) \leq L(X)$ for any X. It's easy to see that if X is a D-space, and $\mathcal{U}$ is an open cover with no subcover of cardinality $< \kappa$, then there must be a closed discrete subset of X of size κ; hence $e(X) = L(X)$. Since closed subspaces of D-spaces are D [**16**], we have the following folklore result:

PROPOSITION 2.1. If X is a D-space, then $e(Y) = L(Y)$ for any closed subspace Y.

The result about countably compact D-spaces being compact is a corollary. Actually, a little bit stronger conclusion holds: if there is an open cover with no subcover of cardinality less than $L(Y)$, then the supremum in the definition of $e(Y)$ is attained, i.e., there is a closed discrete subset of cardinality $e(Y) = L(Y)$. This gives the following fact: a stationary subset S of a regular uncountable cardinal κ is not D.

A useful example to know, because it illustrates some of the limits of what properties could imply D, is an old example due to van Douwen and H. Wicke, which they denoted by Γ.

EXAMPLE 2.2. [**26**] There is a space Γ which is non-Lindelöf and has countable extent, and is Hausdorff, locally compact, locally countable, separable, first-countable, submetrizable, σ-discrete, and realcompact.

This space is not D because $e(X) = \omega < L(X)$.

For a time, all known examples of non-D-spaces failed to be D because the conclusion of 2.1 failed. This was noted by W. Fleissner, essentially bringing up the question (see [**39**]): Is X a D-space iff $e(Y) = L(Y)$ for every closed subspace Y? R. Buzyakova [**14**] asked a related question: Is X hereditarily D iff $e(Y) = L(Y)$ for every $Y \subset X$? Fleissner's question was first answered consistently in the negative by T. Ishiu, who obtained counterexamples both by forcing [**38**] and then from $\diamondsuit^*$ [**39**].

EXAMPLE 2.3. It is consistent that there is a Hausdorff locally countable locally compact non-D-space topology on ω_1 such that $e(Y) = L(Y)$ for every closed subspace Y.

Ishiu's space in [**38**] is constructed so that any uncountable closed set Y contains an uncountable closed discrete set, hence $e(Y) = L(Y)$. The space is made not a D-space by constructing a neighborhood assignment N such that both D and $\cup N(D)$ are nonstationary for any closed discrete set D.

Ishiu [**39**] showed that spaces like his do not exist under $MA + \neg CH$:

Theorem 2.4. (MA + $\neg CH$) Let X be a locally compact locally countable space of size $< 2^\omega$. Then either there is a closed subset F such that $e(F) < L(F)$ or X is a D-space.

Indeed, this follows rather easily from Balogh's result [**9**] that under MA + ¬CH, any locally compact, locally countable space of cardinality $< 2^\omega$ is either σ-closed discrete, or contains a perfect preimage of the ordinal space ω_1.

Ishiu's examples are obtained from certain club guessing sequences. He also showed that under PFA, no example similarly defined from a club guessing sequence (locally compact or not) can be a counterexample to Fleissner's question (see [**39**], Proposition 5.5).

Later, P. Nyikos found a ZFC example of a non-D-space in which $e(Y) = LY)$ for every subspace Y, closed or not, thus answering Buzyakova's question. Nyikos's space is also locally countable and locally compact. Recall that the *interval topology* on a tree T is the topology whose base is the collection of intervals of the form $(s, t] = \{u \in T : s < u \leq t\}$.

Example 2.5. [**47**] Let S be a stationary co-stationary set in ω_1, and let T be the tree of all closed-in-ω_1 subsets of S, ordered by end-extension. Give T the interval topology. Then T is not D, yet $e(Y) = l(Y)$ for every subspace Y.

The neighborhood assignment witnessing not D is the obvious one: $N(x) = \{y : y \leq x\}$. One observes that any closed discrete D such that $N(D)$ covers X must be cofinal in T, that a closed discrete cofinal set can be written as a countable union of antichains, and then uses the known fact that T is Baire (in the sense of the topology generated by $\{t^+ : t \in T\}$, where $t^+ = \{s : s \geq t\}$) to get a contradiction.

Nyikos shows that T contains no Aronszajn subtrees, and from that argues that any uncountable subset of T contains an antichain of the same cardinality. It follows that $e(Y) = L(Y)$ for any subspace Y.

3. Covering properties

Why isn't it easy to prove that Lindelöf spaces (for example) are D? Given a neighborhood assignment N, it is natural to consider a countable cover $\{N(x_n) : n \in \omega\}$ of X. If it could be arranged that $x_n \notin \bigcup_{i<n} N(x_i)$ for each n, then $D = \{x_n : n \in \omega\}$ would be closed discrete and $N(D)$ would cover X. But arranging this is the problem. What about paracompact spaces? One can consider a locally finite open refinement $\mathcal{U}$ of $\{N(x) : x \in X\}$. If for each $U \in \mathcal{U}$, one could find a point $x(U) \in U$ such that $U \subset N(x(U))$, then the $x(U)$'s would be closed discrete and witness the D property. But it may be that the only $N(x)$'s which contain U are for x not in U.

In their article in Open Problems in Topology II, M. Hrušák and J.T. Moore [**36**] list twenty central problems in set theoretic topology; the question whether or not Lindelöf spaces are D is Problem 15 on this list, and is attributed to van Douwen. T. Eisworth's survey article [**28**] in the same volume lists several related questions. In fact, it is not known if any of the following covering properties, even if you add "hereditarily", imply D:

Lindelöf, paracompact, ultraparacompact, strongly paracompact, metacompact, metalindelöf, subparacompact, submetacompact[2], submetalindelöf, paralindelöf, screenable, σ-metacompact.

Since the non-D-space Γ mentioned in the previous section is σ-discrete, it follows that weakly submetacompact does not imply D. But the following question is open:

QUESTION 3.1.

(1) Let X be a countably metacompact space. Must X be a D-space if it is weakly submetacompact? σ-metrizable? [**3**];
(2) Is there a ZFC example of a normal weakly submetacompact non-D-space? [**47**]

There is a consistent example answering 3.1(2): de Caux's σ-discrete Dowker space [**21**] is not D because $e(X) = \omega < L(X)$. I don't know if Balogh's Dowker space [**10**] is or could be made not D.

There is a covering property implied by D: irreducibility. Recall that a space X is *irreducible* if every open cover $\mathcal{U}$ has a minimal open refinement $\mathcal{V}$, i.e., each $V \in \mathcal{V}$ contains a point not in any other $V' \in \mathcal{V}$. Observe that an open cover $\mathcal{U}$ has a minimal open refinement iff there is a closed discrete set D and function $\theta : D \to \mathcal{U}$ such that $d \in \theta(d)$ for each $d \in D$ and $\{\theta(d) : d \in D\}$ is a cover. It is obvious from this that D-spaces are irreducible.

The D property is closed hereditary [**16**], but irreducibility is not. E.g., $2^{\omega_1} \setminus \{p\}$, where $p \in 2^{\omega_1}$, is irreducible but not D [**7**] since it contains a closed copy of ω_1. But the following question is unsettled:

QUESTION 3.2. Is X a D-space iff every closed subspace of X is irreducible?

This question is essentially due to A.V. Arhangel'skii, who with Buzyakova introduced a property called aD [**4**], and later [**3**] proved that for T_1-spaces aD is equivalent to every closed subspace being irreducible, and asked if aD is equivalent to D. Since submetalindelöf implies irreducible [**44**], a positive answer to this question would settle almost all of the questions about the relationship of standard covering properties to the D property.

Since X irreducible implies $e(X) = L(X)$, Question 3.2 could be considered a refinement of the question of Fleissner that was answered in the negative by examples of Ishiu and Nyikos (see Section 2). Nyikos's example is not irreducible, so doesn't answer 3.2, but I don't know if Ishiu's is irreducible or not.

We mentioned above the question of Arhangel'skii whether countably metacompact, weakly submetacompact spaces are D. He also asked about aD; this question is (for T_1-spaces) equivalent to the question whether such spaces are irreducible, which was asked by J.C. Smith [**62**] in 1976 and apparently is still open.

See Section 8 on D relatives for more about aD and irreducibility.

Most of the questions about covering properties implying D have a positive answer for spaces that are either scattered or locally D (i.e., every point has a closed

[2]A space X is *submetacompact* (*submetalindelöf*) if for each open cover $\mathcal{U}$, there is a sequence $\mathcal{U}_n$, $n \in \omega$, of open refinements of $\mathcal{U}$ covering X, such that, for any $x \in X$, there is some $n \in \omega$ such that $\{U \in \mathcal{U}_n : x \in U\}$ is finite (countable). Submetacompact (submetalindelöf) spaces are also called *θ-refinable*(*$\delta\theta$-refinable*). *Weakly submetacompact* or *weakly θ-refinable* is defined the same way as submetacompact except that the $\mathcal{U}_n$'s need not be covers. For definitions of the other covering properties in the list or elsewhere in this article, see [**29**] or [**19**].

neighborhood which is D). That paracompact locally D spaces are D is straightforward, and that paracompact scattered spaces are D is an easy induction of the Cantor-Bendixson height. But the same result for weaker covering properties seems to require a more subtle argument. L.-Xue Peng's proof of this for submetacompact spaces is one illustration of the effective use of topological games in this area.

His result follows from a more general result concerning the following game due to Telgársky [**64**]. Let $\mathbb{K}$ be a closed hereditary class of spaces, and X a space. We define the game $G(\mathbb{K}, X)$. There are two players, I and II. Player I begins by choosing a nonempty closed subset A_0 of X such that $A_0 \in \mathbb{K}$. II responds by choosing a closed set $B_0 \subset X \setminus A_0$. At round $n > 0$, I chooses a nonempty closed subset A_n of B_{n-1} such that $A_n \in \mathbb{K}$, and II responds by choosing a closed set $B_n \subset B_{n-1} \setminus A_n$. We say I *wins* the game if $\bigcap_{n\in\omega} B_n = \emptyset$; otherwise II wins. The space X is said to be $\mathbb{K}$-*like* if Player I has a winning strategy in this game. Trivially, $\mathbb{K}$ is contained in the class of $\mathbb{K}$-like spaces.

Now, let $\mathbb{D}$ be the class of D-spaces. Peng showed that $\mathbb{D}$-like spaces are in fact D:

Theorem 3.3. [**49**] Every $\mathbb{D}$-like space is a D-space.

Proof. The argument shows in fact that X is D as long as II has no winning strategy in $G(\mathbb{D}, X)$. Let N be a neighborhood assignment on X. If A_n is I's play in round n, let II choose a closed discrete subset D_n of A_n such that $N(D_n)$ covers A_n, and then let II play $B_n = B_{n-1} \setminus \cup N(D_n)$. This defines a strategy for II. Since the strategy is not winning, there is a play $A_0, B_0, A_1, B_1, \ldots$ of the game such that $\bigcap_{n\in\omega} B_n = \emptyset$. Noting that $B_n = X \setminus \cup N(D_0 \cup D_1 \cup ... \cup D_n)$ and that $D_{n+1} \subset A_{n+1} \subset B_n$, it is easy to check that if $D = \bigcup_{n\in\omega} D_n$, then D is closed discrete and $N(D)$ covers X. □

A space X is said to be $\mathbb{K}$-*scattered* if every closed subset F of X contains a point p with a closed neighborhood (relative to F) in $\mathbb{K}$. Peng proved:

Theorem 3.4. [**50**] Suppose $\mathbb{K}$ is a class of spaces which is closed under closed subspaces and topological sums, and every $\mathbb{K}$-like space is in $\mathbb{K}$. Then every submetacompact $\mathbb{K}$-scattered space is in $\mathbb{K}$.

Note that the class $\mathbb{D}$ of all D-spaces satisfies the hypotheses of the theorem, so any submetacompact $\mathbb{D}$-scattered space is D. Any scattered space and any locally D-space is $\mathbb{D}$-scattered. Hence the following corollary holds:

Corollary 3.5. If X is submetacompact and either scattered or locally D, then X is a D-space.[3]

But I don't know the answer to the following:

Question 3.6.

(1) Are (sub)metalindelöf scattered spaces D?
(2) Does Question 3.2 have a positive answer in the class of scattered spaces?

Another nice illustration of game theory in this area involves the *Menger property*, which says that, given a countable sequence $\mathcal{U}_n$, $n \in \omega$, of open covers of X, one can select for each n a finite subcollection $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{V}_n$ is a

[3]See the paragraph following Corollary 5.4 for discussion of a direct proof (without using games) of this result.

cover. Obviously, σ-compact spaces are Menger. The following is a recent result of L. Aurichi. We give the proof, which uses a nontrivial game characterization of the Menger property due to W. Hurewicz.

Theorem 3.7. [**6**] Menger spaces are D-spaces.

Proof. Suppose X is Menger, and let N be a neighborhood assignment on X. We play a game in which Player I chooses, in each round n, an open cover $\mathcal{U}_n$ closed under finite unions, and Player II responds by choosing $U_n \in \mathcal{U}_n$. Player II wins if $\{U_n : n \in \omega\}$ covers X. Hurewicz [**37**] proved that X is Menger iff Player I has no winning strategy in this game.

Now, let Player I's first play be $\{\cup\{N(x) : x \in F\} : F \in [X]^{<\omega}\}$. If Player II responds with $\cup\{N(x) : x \in F_0\}$, let I then play

$$\{\cup\{N(x) : x \in F_0 \cup F\} : F \in [X]^{<\omega}, F \cap \cup\{N(x) : x \in F_0\} = \emptyset\}.$$

Then similarly, if II's reply is $\cup\{N(x) : x \in F_0 \cup F_1\}$, I plays

$$\{\cup\{N(x) : x \in F_0 \cup F_1 \cup F\} : F \in [X]^{<\omega}, F \cap \cup\{N(x) : x \in F_0 \cup F_1\} = \emptyset\},$$

and so on. This defines a strategy for Player I. X is Menger, so this can't be a winning strategy. Therefore there is some play of the game with I using this strategy such that, if $F_0, F_1, \dots$ code the plays of II, then $\cup\{N(x) : x \in F_0 \cup F_1 \cup ... \cup F_n : n \in \omega\}$ covers X. Let $D = \bigcup_{n\in\omega} F_n$. Then $N(D)$ covers X. Since for each n, we have $F_n \cap \cup\{N(x) : x \in F_0 \cup F_1 \cup ... \cup F_{n-1}\} = \emptyset$, it is easy to check that D is a closed discrete subset of X. Hence X is a D-space. □

Menger implying D gives that certain other Lindelöf spaces are D. It was proved in [**7**] that if a Lindelöf space X can be covered by fewer than $cov(\mathcal{M})$-many compact sets, where $\mathcal{M}$ is the ideal of meager subsets of the real line, then X is D; in fact, as was pointed out in [**6**], under these assumptions X is Menger. Let us note that this result implies that, e.g., there are no absolute examples of Lindelöf non-D-spaces of cardinality $\aleph_1$. But I don't know the answer to the following:

Question 3.8. Is it consistent that every paracompact space of cardinality $\aleph_1$ is a D-space?

A space X is said to be *productively Lindelöf* if its product with any Lindelöf space is Lindelöf. Alster [**5**] showed that under the Continuum Hypothesis(CH), every productively Lindelöf space of weight $\leq \aleph_1$ has the following property, called *Alster* in [**8**] and [**63**]:

- If $\mathcal{G}$ is a cover of X by G_δ-sets and $\mathcal{G}$ contains a finite subcover of each compact set, then $\mathcal{G}$ has a countable subcover.

Every space which is Alster is Menger [**8**] and hence D. The results just mentioned imply the following:

Theorem 3.9. [**8**] Assume CH. Then every productively Lindelöf space of weight $\leq \aleph_1$, in particular, every first countable or separable productively Lindelöf space, is Alster and hence D.

Alster asked if every productively Lindelöf space is Alster, but this remains unsolved. It is also not known if every productively Lindelöf space is D [**8**].

A space X is *indestructibly Lindelöf* (resp., *indestructibly productively Lindelöf*) if X remains Lindelöf (resp., productively Lindelöf) in every countably closed forcing extension. Lindelöf spaces which are scattered or P-spaces are indestructible.

Indestructibly productively Lindelöf spaces are Alster [**8**], hence D. It is not known if every indestructibly Lindelöf space is D [**8**].

If in the definition of the Menger property, we require that each $\mathcal{V}_n$ has cardinality 1, then we obtain the stronger *Rothberger property.* Rothberger spaces are indestructibly Lindelöf [**8**]. Scheepers and Tall [**61**] show that adding $\aleph_1$ Cohen reals to a model makes any ground model Lindelöf space Rothberger (hence D), and that Moore's ZFC L-space [**46**] is Rothberger.

We conclude this section with an interesting forcing result on D-spaces due to Aurichi, L. Junqueira, and P. Larson.

THEOREM 3.10. [**7**] Let T be a tree of height ω.

(1) If X is a D-space, then X remains D after forcing with T;
(2) If X is a Lindelöf space, and every countable subset of T can be refined to an antichain[4], then X becomes D after forcing with T.

4. Generalized metrizable spaces and base properties

One can often show that spaces with an additional structure such as a base property or generalized metric property are D-spaces.

THEOREM 4.1. The following are D-spaces:

(1) semistratifiable spaces (hence Moore, semimetric, stratifiable, and σ-spaces) [**16**](see also [**30**]);
(2) subspaces of symmetrizable spaces [**20**];
(3) strong Σ-spaces (hence paracompact p-spaces) [**13**];[5]
(4) protometrizable spaces (hence nonarchimedean spaces) [**17**];
(5) spaces having a point-countable base [**4**];
(6) spaces having a point-countable weak base [**51**], [**20**];
(7) sequential spaces with a point-countable W-system [**20**] or point-countable k-network [**52**];
(8) spaces with an ω-uniform base [**6**];
(9) base-base paracompact spaces (hence totally paracompact spaces) [**60**](see also [**59**]).

Some of these properties are defined in [**31**]; see the references given above for the definitions of others. We give a rough idea of why these spaces are D. In the first two cases, and the third with some extra refinements, the proof that metrizable spaces are D basically works. Given a neighborhood assignment N, the space naturally divides into countably many pieces X_n, where the neighborhoods in each piece are "large" in some sense. E.g., for symmetrizable spaces, put a point x in X_n if $N(x)$ contains a ball of radius $1/2^n$; for a σ-space, put x in X_n if there is some member F in the n^{th} discrete collection of a σ-discrete network with $x \in F \subset N(x)$. Then well-order the space so that the points of X_0 come first, then X_1, etc. Let x_0 be the least point and if x_α has been defined for all $\beta < \alpha$, let x_α be the least point of $X \setminus \cup\{N(x_\beta) : \beta < \alpha\}$. Continue until the $N(x_\alpha)$'s cover X. If D is the set of x_α's, then D is closed discrete. So X is a D-space.

[4]A subset S of a tree T *can be refined to an antichain* if for each $s \in S$ one can choose $a(s)$ above s in the tree such that $\{a(s) : s \in S\}$ is an antichain.

[5]In [**41**], S. Lin shows that spaces having what he calls a *σ-cushioned (mod k) pair-network* are D, which generalizes (1) and (3) and implies that $\Sigma^{\#}$-spaces are D.

Protometrizable spaces and nonarchimedean spaces have a base $\mathcal{B}$ such that, given any cover $\mathcal{C}$ by members of $\mathcal{B}$, the largest members of $\mathcal{C}$ form a locally finite family. Thus, given a neighborhood assignment N whose range may be assumed to be included in $\mathcal{B}$, the points corresponding to the largest members of the range witness the D property. Somewhat similar are the proofs for *totally paracompact*, which means that every base contains a locally finite subcover, and *base-base paracompact*, which means that there is a base $\mathcal{B}$ such that every subset $\mathcal{C}$ of $\mathcal{B}$ which is a base has a locally finite subcover. Given N, apply the defining property to the collection $\mathcal{C}$ of all members B of $\mathcal{B}$ such that $x \in B \subset N(x)$ for some $x \in X$.

The proof for point-countable base uses a different method. One defines by a transfinite induction countable closed discrete sets D_α as follows. Let $\mathcal{B}$ be a point-countable base for X, and N a neighborhood assignment whose range we may assume to be included in $\mathcal{B}$.

Now suppose D_β has been defined for all $\beta < \alpha$. Let $F_\alpha = X \setminus \cup N(\bigcup_{\beta<\alpha} D_\beta)$. If $F_\alpha = \emptyset$, the induction stops. Otherwise, let M be a countable elementary submodel of $H(\theta)$ for sufficiently large θ, such that M contains X, $\mathcal{B}$, F_α, ... Let $\prec$ well-order M in type ω. Choose $x_{\alpha,0} \in F_\alpha \cap M$. Suppose $x_{\alpha,i}$ has been defined for each $i < n$. If there no point $x \in F_\alpha \setminus \cup_{i<n} N(x_{\alpha,i})$ such that $N(x) \cap \{x_{\alpha,i}\}_{i<n} \neq \emptyset$, then $D_\alpha = \{x_{\alpha,i} : i < n\}$. On the other hand, if there is such a point, let $x_{\alpha,n}$ be such a point in M chosen so that $N(x_\alpha, n)$ is $\prec$-least. If the induction continues for all $n \in \omega$, then $D_\alpha = \{x_{\alpha,n} : n \in \omega\}$.

One may check that not only is D_α closed discrete, but it is also "sticky" in the sense that $x \in F_\alpha$ and $N(x) \cap D_\alpha \neq \emptyset$ implies $x \in \cup N(D_\alpha)$. Note that by the way D_α was defined, it is automatically relatively discrete, and if it is also sticky, it is closed discrete. Stickyness was introduced by Fleissner and A. Stanley [**30**], and the idea was extended in [**32**], as a way to simplify certain D arguments. One also uses stickyness to show that the union of the D_α's is closed discrete as well. So then, to complete the argument, simply continue the inductive construction of the D_β's until F_α as defined above is empty, and let $D = \bigcup_{\beta<\alpha} D_\beta$.

In [**32**], I used the idea of stickyness to argue that in many cases to show that a space is a D-space, one only has to produce a single nonempty closed discrete sticky subset D of an arb itrary nonempty closed subspace F. The advantage of this is that one can ignore the induction on α and concentrate on constructing a single closed discrete $D \subset F$, which can be countable (even finite); note that you don't need $N(D)$ to cover F.

The proofs that "sequential with a point-countable W-system", "sequential with a point-countable k-network", and "ω-uniform base" imply D are suitably modified versions of the argument for point-countable base.

H. Guo and H.J.K. Junnila [**34**] proved that t-metrizable spaces are hereditarily D-spaces. Here, a space (X, τ) is *t-metrizable* if there exists a weaker metrizable topology π on X and an assignment $H \mapsto J_H$ from $[X]^{<\omega}$ to $[X]^{<\omega}$ such that $cl_\tau(A) \subset cl_\pi(\bigcup_{H\in[A]^{<\omega}} J_H)$. Since $C_p(K)$ is t-metrizable for compact K, a corollary is the earlier result of Buzykova [**14**] that $C_p(K)$ is hereditarily D for any compact K. I improved this to:

THEOREM 4.2. [**32**] If X is a Lindelöf Σ-space, then $C_p(X)$ is hereditarily a D-space.

The above result can be viewed as an explanation for the classical result of D.P. Baturov that Lindelöf degree equals extent for subspaces of these $C_p(X)$'s.

Subsequently, V. Tkachuk [**66**] defined the class of *monotonically monolithic* spaces, which includes spaces $C_p(X)$ for X Lindelöf Σ, as well as spaces with a point-countable base, and proved, using Buzyakova's argument as the model, that monotonically monolithic spaces are hereditarily D. And recently Peng [**58**] defined *weakly monotonically monolithic* spaces, a class which besides including monotonically monolithic spaces is also more general than some of the classes in the list in Theorem 4.1 (e.g., 4.1(6)(7)), and proved they are D-spaces.

From Buzyakova's result about $C_p(K)$ it follows that Eberlein compacta are D-spaces. I generalized this to:

THEOREM 4.3. [**32**] Corson compacta are hereditarily D.

Corson compact spaces are monolithic but it is not known if they must be monotonically monolithic [**66**]. Corson compacta are also Fréchet, hence sequential. I mention this because the following question of I. Juhász and Z. Szentmiklóssy is unsolved:

QUESTION 4.4. [**40**] Is every compact hereditarily D-space sequential?

Possibly relevant to this question is the result of M. Tkačenko [**65**] that a compact space which is the union of countably many left-separated spaces (which are well-known to be D) is sequential.

A result van Douwen proved long before his death in 1987, but which didn't appear in print until 1997 [**24**], is that paracompact GO-spaces (i.e., subspaces of linearly ordered spaces) are D-spaces. Since a GO-space is paracompact iff it does not contain a closed subset homeomorphic to a stationary subset of an uncountable regular cardinal, and such subsets are not D (see Section 2), it follows that a GO-space is a D-space iff it is paracompact.

The class of monotonically normal spaces([**35**]; see also [**31**]) is a common generalization of both metrizable and GO-spaces, and like GO-spaces, they are paracompact iff they do not contain a closed subspace homeomorphic to a stationary subset of an uncountable regular cardinal [**11**]. Thus monotonically normal D-spaces are paracompact, which suggests the following question of Borges and Wehrly:

QUESTION 4.5. [**16**] Is every paracompact monotonically normal space a D-space?

A space (X, τ) is *quasimetrizable* if there is a function $g : \omega \times X \to \tau$ such that (i) $\{g(n,x) : n \in \omega\}$ is a base at x. and (ii) $y \in g(n,x) \Rightarrow g(n+1,y) \subset g(n,x)$. If in (ii) we put " $y \in g(n,x) \Rightarrow g(n,y) \subset g(n,x)$" instead, then X is said to be *nonarchimedean quasimetrizable*. Equivalently, X is nonarchimedean quasimetrizable iff X has a σ-interior-preserving base, where a collection $\mathcal{U}$ of open sets is *interior preserving* if for each $x \in X$, $\bigcap\{U \in \mathcal{U} : x \in U\}$ is open.

QUESTION 4.6. Are (nonarchimedean) quasimetrizable spaces D-spaces?

Another class of spaces relevant to generalized metrizable spaces are the so-called "butterfly spaces". Let (M, τ) be a metrizable space. If τ' is a finer topology on M such that each point $p \in M$ has a neighborhood base consisting of sets B such that $B \setminus \{p\}$ is open in the metric topology τ, then (M, τ') is called a *butterfly* *space over* (M, τ). Many examples in the area of generalized metrizable spaces are butterfly topologies over separable metric spaces (e.g., the tangent disc space, bow-tie space, even the Sorgenfrey line). I don't know the answer to the following:

QUESTION 4.7. Is every butterfly space over a separable metrizable space a D-space?

It is not difficult to show that every such butterfly space is subparacompact, so a negative answer would be most interesting.

5. Unions

The van Douwen-Wicke space Γ is σ-discrete but not D, so the D property is not preserved by countable unions. But the following remains open:

QUESTION 5.1. [**3**] Is the union of two (equivalently, finitely many) D-spaces always a D-space?

The answer is positive in some special cases.

THEOREM 5.2. If $X = \bigcup_{i \leq n} X_i$, then X is D if each X_i:

(a) is metrizable [**4**];
(b) is strong Σ, Moore, $\mathbb{DC}$-like, or regular subparacompact C-scattered [**53**](see also [**69**] for strong Σ and [**43**] for C-scattered);
(c) has a point-countable base [**2**].

Recall that a space X is *C-scattered* if every nonempty closed subspace F has a point with a compact neighborhood (relative to F). A space is $\mathbb{DC}$-*like* if I has a winning strategy in the game $G(\mathbb{DC}, X)$, where $\mathbb{DC}$ is the class of all topological sums of compact spaces. (See Section 3 for the definition of $G(\mathbb{K}, X)$.) The class of $\mathbb{DC}$-like spaces includes all subparacompact C-scattered spaces and all spaces which admit a σ-closure preserving cover by compact sets.

The result about finite unions of Moore spaces gives a positive answer to a question in [**2**].

There are also some positive results about infinite unions. Borges and Wehrly [**16**] show that a countable infinite union of closed D-subspaces is D. The following result of Guo and Junnila is more general:

THEOREM 5.3. [**34**] Suppose $X = \bigcup_{\alpha < \lambda} X_\alpha$, where each X_α is D, and for each $\beta < \lambda$, $\bigcup_{\alpha < \beta} X_\alpha$ is closed. Then X is a D-space.

From this, they derive the following corollary.

COROLLARY 5.4. If X has σ-closure-preserving cover by closed D subspaces, then X is a D-space.

Then they use this to obtain a direct proof (without using games) of Peng's result (see Corollary 3.5) that submetacompact locally D spaces are D; note that it then follows by a straightforward induction on height that submetacompact scattered spaces are D.

Arhangel'skii [**3**] asked whether a countably compact space which is a union of countably many D subspaces must be compact. Juhász and Szentmiklóssy gave a positive answer.

THEOREM 5.5. [**40**](see also [**52**]) If X is countably compact and a countable union of D subspaces, then X is compact.

See Section 8 on D relatives for some generalizations of this result.

Finally, we mention that J.C. Martinez and L. Soukup [**43**] showed that a countable or locally finite union of Lindelöf C-scattered spaces is D.

6. Products

The fact that semistratifiable spaces and strong Σ-spaces are D-spaces and are countably productive as well shows that in many special cases, finite and even countable products of D-spaces are D. Also, Borges and Wehrly [**16**] showed that the product of a D-space with a compact space is D. But in general, the D property is not finitely productive.

EXAMPLE 6.1. [**1**] There is a Lindelöf D-space X and separable metric M such that $X \times M$ not D.

Outline of the construction. Van Douwen [**23**] shows that one can put a finer locally compact locally countable topology with countable extent on any subset of the real line $\mathbb{R}$ of cardinality continuum. Do this on a Bernstein set B, and let $X = \mathbb{R}$ with points of B having van Douwen's neighborhoods and points of $\mathbb{R} \setminus B$ having their usual Euclidean neighborhoods. One proves that this space is Lindelöf and D using the fact that any open superset of $\mathbb{R} \setminus B$ contains all but countably many points of $\mathbb{R}$. Now $X \times B$, where B has its usual topology, contains the closed copy $\{(x, x) : x \in B\}$ of B with van Douwen's topology, which is not D as it is not Lindelöf but has countable extent. □

Assuming CH, Borges and Wehrly [**18**] show that there exists a first countable regular hereditarily Lindelöf space Y such that Y^2 is perfectly normal and hereditarily separable but not a D-space. Tall [**63**] obtains consistent examples of Rothberger spaces (for the definition, see the discussion after Theorem 3.9) whose squares are not D; the key idea here is to take known examples of two Lindelöf spaces whose product is not Lindelöf but has countable extent, and note that adding ω_1-many Cohen reals makes ground model Lindelöf spaces Rothberger [**61**] but preserves countable extent.

An interesting specific example is the Sorgenfrey line and its countable powers. In the first published paper on D-spaces, van Douwen and Pfeffer [**25**] prove that the Sorgenfrey line and its finite powers are D-spaces. Later, answering one of their questions, P. de Caux [**22**] proved that every finite power of the Sorgenfrey line is hereditarily D. But the following is still open.

QUESTION 6.2. [**22**] Let S be the Sorgenfrey line. Is S^ω a D-space? Is it hereditarily D?

Since S^ω is hereditarily subparacompact, and it is not known if subparacompact implies D, it would be particularly interesting if the answer to the question is "no".

The D-property in products of scattered spaces and some more general spaces have been investigated. In the previous section, we discussed unions of $\mathbb{DC}$-like and C-scattered spaces. We recall here that any countable product of (sub)paracompact $\mathbb{DC}$-like spaces is (sub)paracompact.

Theorem 6.3. [**55**] A countable product of paracompact $\mathbb{DC}$-like (in particular, C-scattered) spaces is a D-space.

Fleissner and Stanley [**30**] showed that if each X_α is scattered of height 1, then the box product $\square_{\alpha<\kappa} X_\alpha$ is a D-space. They also showed that a subset of a finite product of ordinals is D iff it is metacompact iff it does not contain a closed subset homeomorphic to stationary subset of regular uncountable cardinal. The box product result was improved by Peng [**55**] to scattered of height $\leq k$, where $k \in \omega$.

7. Mappings

There are only a small number of results on mappings of D-spaces. Borges and Wehrly proved the following useful result:

Theorem 7.1. [**16**] If X is a D-space, then so is every closed image of X and every perfect pre-image of X.

Note that fact that the product of a D-space with a compact space is D is a corollary to the result on perfect pre-images.

As was pointed out in [**20**], quotient or even open images of D-spaces need not be D. Any first countable space is the open image of a metrizable space, but of course there are first-countable spaces which are not D-spaces. That spaces with a point-countable base are D (Theorem 4.1(5)) implies that open s-images of metrizable spaces are D (where "s-image" means that fibers are separable). D. Burke extended this with the following result:

Theorem 7.2. [**20**] The quotient s-image of a space with a point-countable base is a D-space.

This also follows from Theorem 4.1(7).

Of course, continuous images of D-spaces need not be D, but the following question is open:

Question 7.3. (Arhangel'skii, see [**13**]) Let X be a Lindelöf D-space. Is every continuous image of X a D-space?

8. D relatives

8.1. Property aD. Arhangel'skii and Buzyakova [**4**] introduced a weakening of the D property called aD. A space X is aD if for each closed $F \subset X$ and for each open cover $\mathcal{U}$ of X, there is a locally finite $A \subset F$ and $\phi : A \to \mathcal{U}$ with $a \in \phi(a)$ and $F \subset \cup\phi(A)$.

It is obvious that D-spaces are aD. In a later paper, Arhangel'skii showed that (for T_1-spaces) the aD property has an equivalence in terms of irreducibility (see Section 3 for the definition of irreducible):

Theorem 8.1. [**3**] A T_1-space X is aD iff every closed subspace of X is irreducible.

So questions about aD become questions about irreducibility. Arhangel'skii asked in [**3**] if every Tychonoff aD space is D. By the previous theorem, this is equivalent to Question 3.2.

J. Mashburn [**44**] proves (for T_1-spaces) that submetalindelöf spaces are irreducible (also weakly $\delta\bar{\theta}$-refinable, which we shall not define here). Z. Yu and Z.

Yun [**69**] improve this by showing that any finite union of submetalindelöf spaces is irreducible.

We mentioned Arhangel'skii's question whether the union of two D-spaces is D. The following question of his is also open and seems more likely to have a positive answer:

QUESTION 8.2. [**3**] If X is the union of two D subspaces, must X be aD (i.e., irreducible)?

8.2. Dually discrete. One can obtain generalizations of D-spaces by weakening the requirement in the definition that D be closed discrete. If P is a property of topological spaces, J. van Mill, Tkachuk, and Wilson [**45**] call a space X *dually P* if for every neighborhood assignment N, there is a subspace D of X having property P such that $N(D)$ covers X. In particular, a space X is *dually discrete* if for any neighborhood assignment N, there is a relatively discrete D with $\cup N(D) = X$.

Dually discrete is a much weaker property than D, as evidenced by the following result.

THEOREM 8.3.

(1) [**54**] Every GO-space is dually discrete;
(2) [**27**] Every tree with the interval topology is dually discrete.

Theorem 8.3(1) answers a question in [**15**] and 8.3(2) a question in [**47**].

Surprisingly, the same questions about whether or not standard covering properties imply D are also unsolved for this weaker version. In particular:

QUESTION 8.4. [**1**] Are hereditarily Lindelöf spaces dually discrete?

Since GO-spaces are dually discrete, so is any ordinal space. Answering a question in [**1**], Peng [**56**] showed that any finite product of ordinals is always dually discrete. But as with the D-property, in general dually discrete is not finitely productive, at least consistently. In [**1**] it is noted that if in Example 6.1 above, the van Douwen line is replaced by the Kunen line (which exists only under CH), then we have a D-space X and a separable metrizable M such that $X \times M$ is not dually discrete. The Kunen line, being hereditarily separable and not Lindelöf, is clearly not dually discrete. One would have a ZFC example if the van Douwen line were (or could be made to be) not dually discrete, but I don't know if that is the case or not.

In [**15**], the authors show that under $\diamondsuit$, $\mathbb{R}^\kappa$ is not dually discrete for $\kappa > \omega$, and ask if this is true in ZFC.

An easy argument shows that every space X is dually scattered. There is however a ZFC example of a scattered space X which is not dually discrete [**15**]: take a right-separated subset of type κ^+ in a space whose hereditary density is κ and hereditary Lindelöf degree is greater than κ (such spaces were constructed in ZFC by Todorcevic [**67**]).

Here are some more questions from [**15**]:

QUESTION 8.5. Do any of the following imply dually discrete:

(1) Monotonically normal;
(2) σ-(relatively) discrete;
(3) dually metrizable?

8.3. Linearly D and transitively D. Putting conditions on the neighborhood assignment yields generalizations of the D property. Guo and Junnila define X to be *linearly* D if for every neighborhood assignment N whose range $\{N(x) : x \in X\}$ is a linearly ordered (by $\subseteq$) collection of sets, then there is a closed discrete set D such that $X = \cup N(D)$.

THEOREM 8.6. [**33**]
(1) X is linearly Lindelöf iff X is linearly D and $e(X) = \omega$;
(2) Submetalindelöf spaces are linearly D;
(3) Finite unions of linearly D spaces are linearly D;
(4) Countably compact spaces which are the union of countably many linearly D subspaces are compact.

In [**54**], Peng had a different definition of linearly D; his property is now called "transitively D". A space X is *transitively* D if for every neighborhood assignment N such that $y \in N(x)$ implies $N(y) \subset N(x)$, there is a closed discrete set D such that $X = \cup N(D)$.

Peng proved the following:

THEOREM 8.7. [**57**]
(1) Every transitively D space is linearly D;
(2) Metalindelöf spaces are transitively D;[6]
(3) Finite unions of transitively D spaces are transitively D.

It follows from Theorem 8.6(1) that any linearly Lindelöf non-Lindelöf space is linearly D but not D. But the following is unsettled:

QUESTION 8.8. [**57**] Is there a transitively D space which is not D?

I don't know if any of the known examples of linearly Lindelöf non-Lindelöf spaces[7] are transitively D. Nyikos's space (Example 2.5) is not transitively D since the neighborhood assignment witnessing not D is transitive. Van Douwen-Wicke's space Γ (Example 2.2) is not even linearly D since $e(\Gamma) = \omega$ but Γ is not linearly Lindelöf. I don't know if Ishiu's examples (Example 2.3) are transitively D, or if Nyikos's space is linearly D.

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