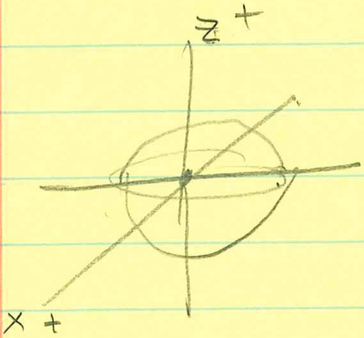
SECTIONS 12.1–12.4.

§12.1 • Vectors + coord systems -



– Right hand rule
– Drawing / Visualizing
e.g. $x^2+y^2+z^2=1$
sph, radius 1
center at $(0,0,0)$

★ ((sphere $= (x-a)^2+(y-b)^2+(z-c)^2 = r^2$))

• Distance between 2 pts $(x_1, \ldots, x_n) = \bar{x}$
$(y_1 \ldots y_n) = \bar{y}$
is $\sqrt{\sum_i (x_i-y_i)^2}$ (pythagorean theorem).

$\boxed{\text{• class}}$: Distance from $(1,2,3)$ to $(5,6,7)$
$= \sqrt{4^2+4^2+4^2} = 4\sqrt{3}$ //.

$\boxed{\text{• class (22)}}$

Find eqn of sphere if $\overset{P_1}{(5,4,3)}, \overset{P_2}{(1,6,-9)}$ is a line of diameter.

(1) Picture

$m = \frac{P_1+P_2}{2} = (3,5,-3)$

$r = |P_2 - m| = \sqrt{(-2)^2+1^2+(-9-3)^2} = \sqrt{4+1+36} = \sqrt{41}$

$\Rightarrow (x-3)^2 + (y-5)^2 + (z-(-3))^2 = 41$

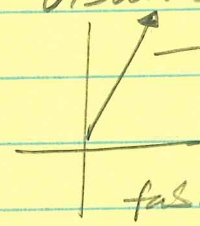
| class 40 | • find equations defining solid upper hemisphere, r=2, center at origin.

$\boxed{\cdot\ x^2+y^2+z^2 \le 4, \ \cdot z \ge 0}$

§ 12.2 Vectors : a directed quantity

Speed is just a quantity, velocity is a vector.

Write a vector as $\langle a,b\rangle$, visualize as tail at $(0,0)$, head at (a,b), e.g.

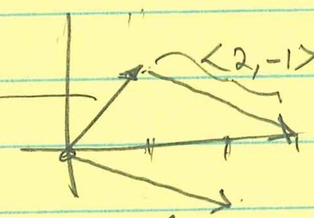


scale, add, subtract in obvious fashion:

$$\langle a,b\rangle + \langle c,d\rangle = \langle a+c, b+d\rangle, \quad k\langle a,b\rangle = \langle ka, kb\rangle$$

Pictorially adding vectors = put tail of one at head of second.

e.g. $\langle 1,1\rangle + \langle 2,-1\rangle = \langle 3,0\rangle$



Notice $\langle a,b\rangle + \langle c,d\rangle = \langle c,d\rangle + \langle a,b\rangle$.

entries are components

Length of $\bar{v} = \langle v_1, \ldots, v_n\rangle = \sqrt{\sum v_i^2}$

See P 802 for properties of vectors.

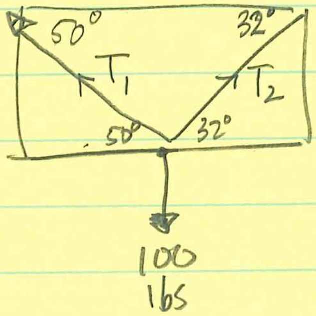
In $\mathbb{R}^3$, $\vec{i} = \langle 1\,0\,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, $\vec{k} = \langle 0\,0,1\rangle$

vector is unit vector if $|\bar{v}| = 1$

- Class
  - Compute length of $\bar{v} = \langle 1,2,3\rangle$
  - Find unit vector $\vec{u}$ in same direction

SOLN $|\bar{v}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, $\bar{u} = \bar{v}/|\bar{v}|$ ∎

Application (Example 7, p 804)



Find T_1, T_2.

Break into components:

$$T_1 = -|T_1| \cos 50 \, \vec{i} + |T_1| \sin 50 \, j$$

$$T_2 = |T_2| \cos 32 \, i + |T_2| \sin 32 \, j$$

Ask class What info is not used?

Answer: Force $\uparrow$ = 100 lbs

Force $\rightarrow$ = 0

So:

$$|T_1| \cos 50 = |T_2| \cos 32 \Rightarrow \boxed{|T_1| = \frac{\cos 32}{\cos 50} |T_2|}$$

$$|T_1| \sin 50 + |T_2| \sin 32 = 100$$

$$\frac{\cos 32}{\cos 50} |T_2| \sin 50 + |T_2| \sin 32 = 100$$

$|T_2| \approx 65. \Rightarrow |T_1| \simeq 85.$ ∎

Class: Find unit vector // to tangent line to $y = x^2$ at $(2,4)$.

Soln: tan line has slope $y' = 2x$ (at $(2,4)$) $= 4$.

so vector is $\langle 1, 4 \rangle$ (1 over, 4 up)

unit = divide by $\sqrt{1^2 + 4^2} = \sqrt{17} \Rightarrow \dfrac{\langle 1, 4 \rangle}{\sqrt{17}}$ ∎

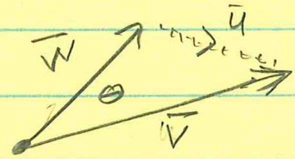
§ 1.3. (First actual new tool) Dot product.

DEF $\bar{v} = \langle v_1 \ldots v_n \rangle$, $\bar{w} = \langle w_1 \ldots w_n \rangle$, $\bar{v} \cdot \bar{w} = \sum_i^n v_i w_i$

Facts ① $|v| = \sqrt{v \cdot v}$ or $\bar{v} \cdot \bar{v} = |v|^2$.

② $\bar{v} \cdot \bar{w} = |v||w| \cos\theta$, where

SUPER USEFUL



Proof $\bar{u} + \bar{w} = \bar{v}$

Law of cosines $|u|^2 = |v|^2 + |w|^2 - 2|v||w| \cos\theta$

$$(v-w)\cdot(v-w) = v\cdot v \quad w \cdot w$$

$$v\cdot v - 2 v\cdot w + w\cdot w = v\cdot v + w \cdot w - 2|v||w|\cos\theta$$

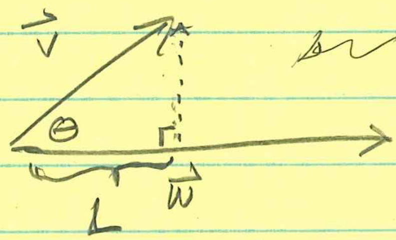
$$v\cdot w = |v||w| \cos\theta. \quad \blacksquare$$

Class find $\angle$ between $\langle 1,1,1,1 \rangle = v$ and $\langle 1,2,3,4 \rangle = w$

soln $v \cdot w = 10$, $|v| = 2$, $|w| = \sqrt{30}$

$$10 = 2\sqrt{30} \cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{10}{2\sqrt{30}}\right) \blacksquare$$

Hardest part of today! Projection formula



so break $\bar{v}$ into a piece $\parallel \bar{w}$ and " $\perp \bar{w}$ (;)

piece $\parallel$ to $\bar{w}$ is just $L \cdot \frac{\vec{w}}{|w|}$, and L is just $|v| \cos\theta$. But $|w||v|\cos\theta = \bar{w} \cdot \bar{v}$

so $L = \frac{\bar{w} \cdot \bar{v}}{|w|}$

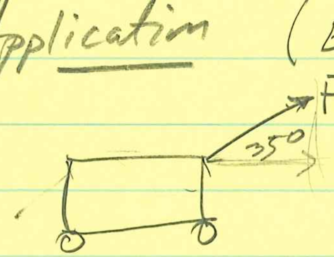
$\Rightarrow$ piece $\parallel$ is $\left(\frac{w \cdot v}{w \cdot w}\right)\vec{w}$ $\blacksquare$

Def $\boxed{\mathrm{Proj}_{\overline{w}} \overline{v} = \frac{v \cdot w}{w \cdot w} \vec{w}}$

Class (Example 6) Find $\mathrm{proj}_{\langle -2, 3, 1\rangle} \langle 1, 1, 2\rangle$ $\left(= \frac{3}{14} \langle -2, 3, 1\rangle \right)$ SOLN

Nice Application (Example 7) Work = force $\times$ distance

$W = F \cdot D$



Wagon pulled 100 m along path using 70 N force. compute work

SOLN $F = 70 \cos 35\, i + 70 \sin 35\, j$

$D = 100\, i + 0\, j$

$F \cdot D = 7000 \cos 35 = 5734\ N/m$ ∎ Joules

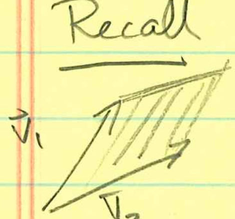
§ 12.4 Cross Product. (3D specific)

$\overline{v}_1 = ai + bj + ck = \langle a, b, c\rangle$

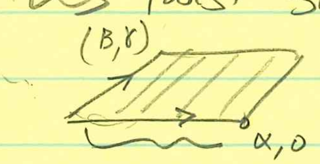
$\overline{v}_2 = di + ej + fk = \langle d, e, f\rangle$

Def $\overline{v}_1 \times \overline{v}_2 = \det \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} = i \begin{vmatrix} b & c \\ e & f \end{vmatrix} - j \begin{vmatrix} a & c \\ d & f \end{vmatrix} + k \begin{vmatrix} a & b \\ d & e \end{vmatrix}$

[**Recall** $\begin{vmatrix} x & y \\ z & w \end{vmatrix} = xw - yz$ ← measures area



⇝ twist so $v_2 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$



$\begin{vmatrix} \alpha & \beta \\ 0 & \gamma \end{vmatrix} = \alpha\gamma$ = base · height = area]

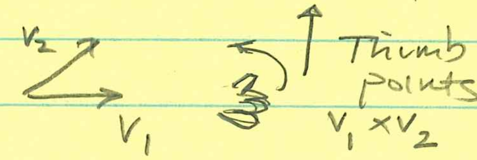
$\boxed{\text{Notice}}$ $\vec{v} \cdot \vec{w}$ = scalar. $\overline{v} \times \overline{w}$ = vector.

Theorem $V_1 \times V_2 \perp V_1, V_2$

Proof: $\left[i(bf-ce) - j(af-cd) + k(ae-bd)\right] \cdot \left[ai + bj + ck\right]$

$= \underbrace{a(bf-ce) - b(af-cd)}_{\text{cancel.}} + c(ae-bd)$

Facts
- $V_1 \times V_2 = -V_2 \times V_1$
- Right hand rule ⇝
- $|V_1 \times V_2| = |V_1||V_2| \sin\theta$

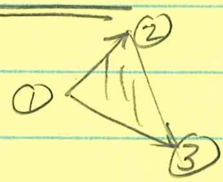


$\boxed{\text{Class}}$ Find vector $\perp$ to $\langle 1,1,1\rangle, \langle 1,2,3\rangle$

Soln: $\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = i - 2j + k$. **check** $\langle 1 1 1\rangle \cdot \langle 1,-2,1\rangle = 0$ ☺

$\langle 1 2 3\rangle \cdot \langle 1,-2,1\rangle = 0$

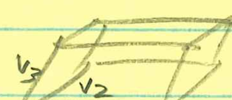
Application: Area of Δ with vertices ① $(1,-1,1)$, ② $(2,0,2)$, ③ $(2,1,4)$



$\bar{v} = \vec{12} = \langle 1,1,1\rangle$ Area $= \frac{1}{2}$ // ogram

$\bar{w} = \vec{13} = \langle 1,2,3\rangle$ $= \frac{1}{2}|v||w|\sin\theta = \frac{1}{2}\left|\langle 1,-2,1\rangle\right| = \frac{\sqrt{6}}{2}$

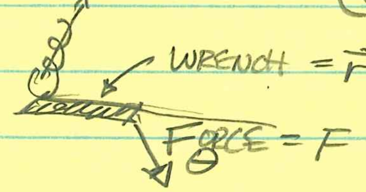
Applicatn Volume of //piped



$= \text{base} \cdot \text{height} = |V_1 \times V_2| \cdot |V_3| \cos\theta$

$= (V_1 \times V_2) \cdot V_3$!

Example 6 **Applicatn** Torque



$\tau = \vec{r} \times \vec{F}$

Notice — Torque is $\perp$ to r, F ✓

$F = 40N$, $r = .25\ M$ $\theta = 75^0$

$r = .25\, j$, $F = \underbrace{40\cos 75}_{\alpha}\, j - \underbrace{40 \sin 75}_{\beta}\, \bar{k}$

$\begin{vmatrix} i & j & k \\ 0 & 1/4 & 0 \\ 0 & \alpha & \beta \end{vmatrix} = \frac{1}{4}\beta\, \vec{i}$

Find τ