SECTION 13.3, 13.4.

WARM UP FIND UNIT TANGENT VECTOR @ $t=0$ to

$$\vec{r}(t) = (\tan^{-1}(t),\ 2e^{2t},\ 8te^{t}).$$

$x = \tan^{-1} y$

$\left[\frac{d}{dt}(\tan^{-1}(t)) = \frac{1}{1+t^2}\right] \leftarrow$ Put on board

$\sqrt{x^2+1}$, x, 1, y

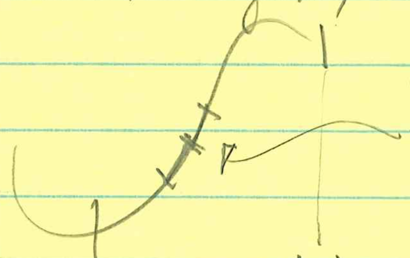
$\tan(y) = x$

$\sec^2 y\, dy = dx$

$\Delta\ \frac{1}{1+x^2} = \cos^2 = \left(\frac{1}{\sqrt{x^2+1}}\right)^2 \Longleftarrow dy/dx = \cos^2 y = \cos^2(\tan^{-1} x)$

$\Rightarrow \vec{r}'(t) = \left(\frac{1}{1+t^2},\ 4e^{2t},\ 8te^{t}+8e^{t}\right)$ @ $t=0 \Rightarrow \langle 1,4,8\rangle/\sqrt{101}$.

§ 13.3 Arclength, curvature, TNB.



$t=a$ $\quad$ $t=b$

ds, dy, dx

$\frac{ds}{dt} = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} \Rightarrow \sum ds = \sum \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\Rightarrow \text{Arclength} = \int_{t=a}^{b} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Example: find length of helix $(\cos t, \sin t, t)$, $t = (0 .. 2\pi)$

$$= \int_{t=0}^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt = \sqrt{2}\int_0^{2\pi} t = \sqrt{2}\cdot 2\pi$$

$\boxed{\text{DEF}}$ $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = |\vec{r}'(t)|$, $s(t) = \int_a^t |\vec{r}'(t)|\, dt$. arclength

By F.T.C. $ds/dt = |\vec{r}'(t)|$

DRAW PIX

$\boxed{\text{big radius} \ \text{“} \ \text{small } \kappa}$

Unit tangent vector $\vec{T}(t) = \vec{r}'(t)/|r'(t)|$

$$\kappa = \text{curvature} = \left|\frac{dT}{ds}\right| = \left|\frac{dT/dt}{ds/dt}\right| = \frac{|T'(t)|}{|\vec{r}'(t)|}$$

Example Circle radius a: $(a\cos t, a\sin t) = \vec{r}$, $\vec{T} = (-\sin t, \cos t)$ $\Rightarrow \frac{|T'|}{|r'|} = \frac{1}{a}$

$\Rightarrow \vec{r}' = (-a\sin t, a\cos t)$, $|\vec{r}'(t)| = a$

REMEMBER!

FROM LAST CLASS: $\boxed{K(t) = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|r'(t)|^3}}$

LAST CLASS, WE COMPUTED κ FOR $\vec{r}(t) = (t, t^2, t^3)$.

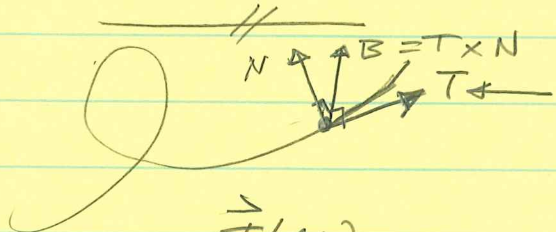
Class Find κ for helix $(\cos t, \sin t, t) = \vec{r}(t)$

SOLN $r' = (-\sin t, \cos t, 1)$ $\quad r' \times r'' = \begin{vmatrix} i & j & k \\ -s & c & 1 \\ -c & -s & 0 \end{vmatrix} = \langle \sin t, -\cos t, 1 \rangle$

$r'' = (-\cos t, -\sin t, 0)$

$\Rightarrow |r'| = \sqrt{2}, \ |r' \times r''| = \sqrt{2} \Rightarrow \boxed{\kappa = 1/2}$.

TNB frame



$T \leftarrow \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ (Know already)

N = Normal vector $= \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$, $B = T \times N$ (Normalized accelleration)

notice $B \perp T, N$

Class Compute T, N, B for $\vec{r}(t) = (\cos t, \sin t, t)$.

$\vec{T} = \dfrac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}}$, $\vec{N} = \dfrac{1}{\sqrt{2}} \dfrac{\langle -\cos t, -\sin t, 0 \rangle}{1} \Big/ \dfrac{1}{\sqrt{2}} = \langle -\cos, -\sin, 0 \rangle$

$\vec{B} = \dfrac{1}{2} \begin{vmatrix} i & j & k \\ -s & c & 1 \\ -c & -s & 0 \end{vmatrix} = \dfrac{1}{\sqrt{2}} \langle -\sin t, -\cos t, -1 \rangle$

CHECK
$B \cdot T = \frac{1}{2}(-s^2 - c^2 + 1) = 0$ ✓
$B \cdot N = (sc - sc) = 0$ ✓

Class 23. Find κ for $\vec{r}(t) = (3t, 4\sin t, 4\cos t)$

① $\vec{r}' = (3, 4\cos t, -4\sin t)$ $\quad \vec{r}' \times \vec{r}'' = \begin{vmatrix} i & j & k \\ 3 & 4c & -4s \\ 0 & -4s & -4c \end{vmatrix} = \langle -16, 12c, -12s \rangle$

$\vec{r}'' = (0, -4\sin t, -4\cos t)$

So $|r' \times r''| = \sqrt{(16)^2 + 144(c^2+s^2)} = \sqrt{256+144} = 20$

$|r'| = \sqrt{9 + 16(s^2+c^2)} = 5$ $\qquad \Rightarrow \kappa = \dfrac{20}{5^3} = \boxed{4/25}$

13.3 - 4

§ 13.4 If $\vec{r}(t)$ is position, $\vec{r}' =$ velocity, $\vec{r}'' =$ accel.

How about reversing things?

Suppose $\vec{r}(0) = \langle 1,0,0 \rangle$, $\vec{v}(0) = \vec{r}'(0) = \langle -1,-1,1 \rangle$

$\vec{r}''(0) = \vec{a}(t) = \langle 4t, 6t, 1 \rangle$

Find $\vec{r}'(t)$, $\vec{r}(t)$. Soln gotta integrate.

$$\vec{r}'(t) = \int \vec{r}''(t)\,dt = \underbrace{\langle 2t^2, 3t^2, t \rangle}_{} + \langle c_1, c_2, c_3 \rangle$$

C

Know $\vec{r}'(0) = \langle -1,-1,1 \rangle =$ PLUG IN 0 $\langle 0,0,0 \rangle + \langle c_1, c_2, c_3 \rangle$

$\Rightarrow \vec{r}'(t) = \langle 2t^2-1, 3t^2-1, t+1 \rangle$

$\vec{r}(t) = \int \vec{r}'(t) = \langle \frac{2}{3}t^3 - t, t^3 - t, \frac{t^2}{2} + t \rangle + \langle d_1, d_2, d_3 \rangle$

PLUG IN 0

$\langle 1,0,0 \rangle = \vec{r}(0) = \langle 0, 0, 0 \rangle + \langle d_1, d_2, d_3 \rangle$

$\Rightarrow \vec{r}(t) = \langle \frac{2}{3}t^3 - t + 1, t^3 - t, \frac{t^2}{2} + t \rangle$ ∎

(TO CHECK: Differentiate + plug in consts.)

Main Topic: {Breaking $\vec{a}$ = accelleration $= \vec{r}''(t)$ into Components with respect to T, N.}

$\vec{a} = a_T \vec{T} + a_N \vec{N}$ How get a_T, a_N?

① Project $\vec{r}''$ onto $\vec{T}$: $a_T = \dfrac{\vec{r}'' \cdot \vec{T}}{|T| \leftarrow 1} = \vec{r}'' \cdot \underbrace{\dfrac{\vec{r}'}{|\vec{r}'|}}_{T}$

So a_T is easy to get.

Since $|T| = 1 = |N|$, $|\vec{r}''|^2 = a_T^2 + a_N^2$ so could get a_N this way.

13.3-4

When we work this out, we find $a_N = \dfrac{|\bar r' \times \bar r''|}{|\bar r'|} = \boxed{\kappa\, |r'|^2}$

$\boxed{\text{Memorize}}$ $a_T = \dfrac{\bar r' \cdot \bar r''}{|r'|}$, $a_N = \dfrac{|r' \times r''|}{|r'|}$. N comes before T
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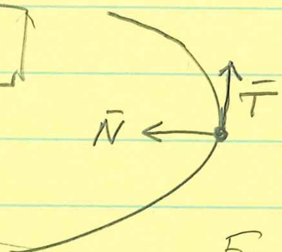
$\boxed{\text{Intuition}}$



Sharp turn in the road. $\bar F = m\bar a$

$a_T T + a_N N$

because $a_N = \kappa |r'|^2$ $\longrightarrow$ Force has a big N componant, proportional to speed^2!

$\Rightarrow$ double speed, hit door $4\times$ as hard!

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$\boxed{\text{Class}}$ compute a_T, a_N for $(t, t^2, t^3) = \bar r(t)$

$\boxed{\text{Soln}}$ $\bar r'(t) = (1, 2t, 3t^2)$ $\quad r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2\rangle$

$\bar r''(t) = (0, 2, 6t)$

$|\bar r'(t)| = \sqrt{1+4t^2+9t^4}$
$\bar r' \cdot \bar r'' = 4t + 18t^3$
$\Rightarrow a_T = \dfrac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}$, $a_N = \dfrac{\sqrt{36t^4+36t^2+4}}{\sqrt{1+4t^2+9t^4}}$

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17. $\vec a(t) = (2t, \sin t, \cos 2t)$, $\vec V(0) = (1 0 0)$, $r(0) = (0 1 0)$

Find $\bar r(t)$.

Soln: $\vec v(t) = \int \vec a(t) = \left(t^2, -\cos t, \frac{\sin 2t}{2}\right) + (c_1, c_2, c_3)$

@ 0: $(0, -1, 0) + (c_1, c_2, c_3) = 1 0 0$

$\Rightarrow (c_1, c_2, c_3) = (1 1 0) \Rightarrow$

$\bar r(t) = \int \left(t^2+1, 1-\cos t, \frac{\sin 2t}{2}\right) = \left\langle \frac{t^3}{3}+t, t-\sin t, -\frac{\cos 2t}{4}\right\rangle + (d_1, d_2, d_3)$

" at $t=0$

$\bar r(t) = \left(\frac{t^3}{3}+t, t+1-\sin t, \frac{1-\cos 2t}{4}\right) \Longleftarrow \left(0, 1, \frac{1}{4}\right) = (d_1, d_2, d_3) \Longleftarrow (0 1 0)$