§ 14.4

WARM UP

(41) $w = f(x,y,z) = \dfrac{y}{x+y+z}$ find $f_y(2,1,-1)$

(A) $\dfrac{\partial w}{\partial y} = \dfrac{(x+y+z)\cdot\left(\frac{\partial}{\partial y}(y)\right) - (y)\left(\frac{\partial}{\partial y}(x+y+z)\right)}{(x+y+z)^2} = \dfrac{(x+y+z) - y}{(x+y+z)^2} = \dfrac{x+z}{(x+y+z)^2}$

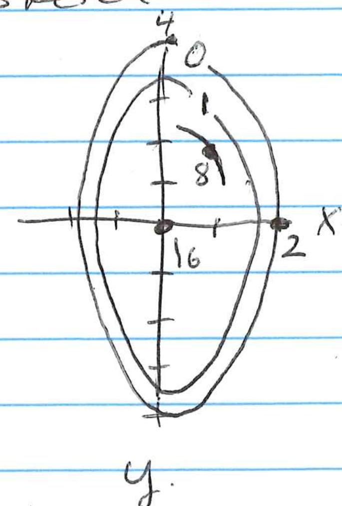
(bottom · d(top) − top d(bottom)) / bottom^2.

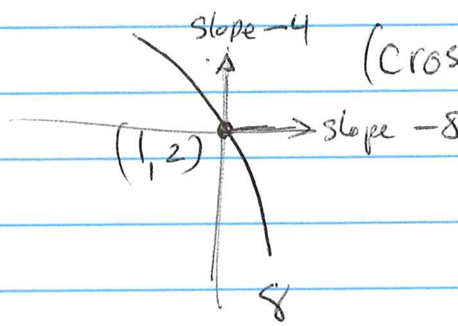
(11) $f(x,y) = 16 - 4x^2 - y^2$

find $f_x(1,2)$, $f_y(1,2)$ and interpret as slopes, sketch.

(A) $f_x = -8x \ @ \ (1,2) = -8$
$f_y = -2y \ @ \ (1,2) = -4$.

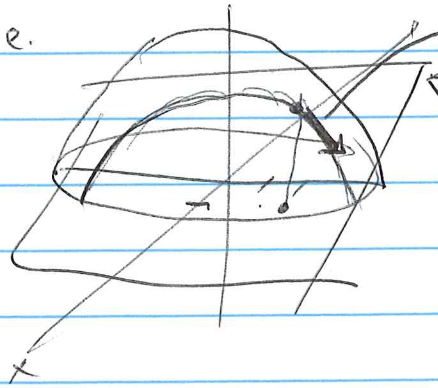
Contours





(cross sections.)

i.e.



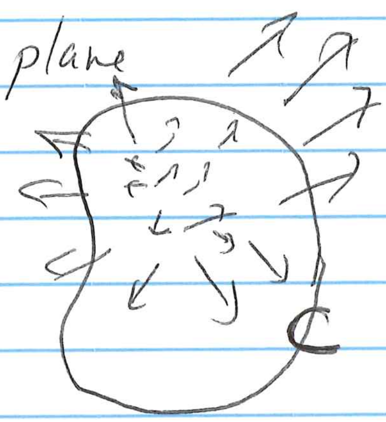
same pix w/ slice where y = 2 will have −8 Δ in z.

Why do we care about tangent planes, partial derivs, etc?

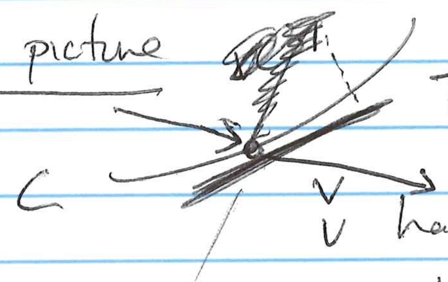
A: Big Thms are Green, Gauss, Stokes.

Green: vector field in plane + a closed curve C

Compute flow across C.



Local picture



V has two components: one $\perp$ to T, one $\parallel$ to T

intuition flow across C "=" part of V $\perp$ to T_C.

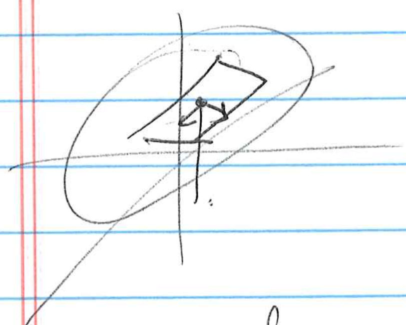
§ 14.4 Tangent planes and linear approx.

Continuing w/ our example

ME $z = f(x,y) = 16 - 4x^2 - y^2$. Find tangent plane @ $(1,2) = (x,y)$

$P = (1, 2, f(1,2)) = 16 - 4(1)^2 - (2)^2 = 16 - 4 - 4 = 8$

$p = (1, 2, 8)$. $\langle x-1, y-2, z-8 \rangle \cdot \vec{n} = 0$.



Last time: Normal vect @ p

$= (1, 0, f_x(p)) \times (0, 1, f_y(p))$

$f_x(p) = -8$, $f_y(p) = -4$ So

$$\begin{vmatrix} i & j & k \\ 1 & 0 & -8 \\ 0 & 1 & -4 \end{vmatrix} = 8i + 4j - k = \langle 8, 4, -1 \rangle$$

$\Rightarrow T = \boxed{8(x-1) + 4(y-2) - (z-8) = 0}$

CLASS $z = xe^{xy}$. Find tangent plane at $(x,y) = (2,0)$

Soln: $f(1,0) = 2e^0 = 1 \Rightarrow p = (2, 0, 1)$.

$f_x = x(ye^{xy}) + e^{xy} \cdot 1$ @ $(2,0) = 1$

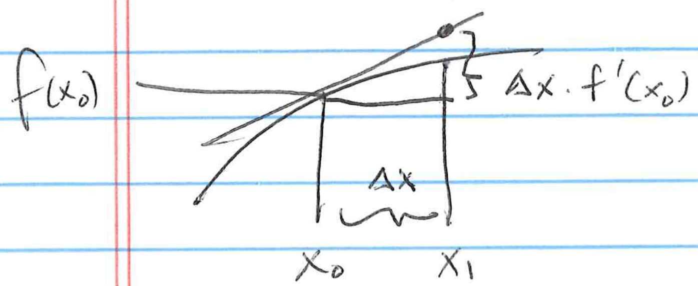
$f_y = x^2 e^{xy}$ @ $(2,0) = 4$.

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 4 \end{vmatrix} = -i - 4j + k$$

$\boxed{\Rightarrow T_{(2,0,1)} = (x-2, y, z-1) \cdot (-1, -4, 1) = 0}$

$\boxed{\text{Next topic}}$ linear approximations.

Recall last time we saw Newtons Method



$$\underline{f(x_1) \approx f(x_0) + \Delta x \cdot f'(x_0)}$$

Approximate function of 2 vars the same way.

$\boxed{\text{DEF}}$ $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

NOTICE THIS is $\boxed{\text{EXACTLY}}$

$$[z - f(a,b)] = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= \cancel{\langle 1, -f_y, -f_x \rangle \cdot (x-a}$$

$$\boxed{-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1(z-f(a,b)) = 0}$$

If wanted a $\boxed{\text{quadratic}}$ approx,

CALC 1

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \cdots$$

There is a Multivar. version of $\boxed{\text{Taylor}}$ ↗

~~THEOREM~~ DEF $z = f(x,y)$ is $\boxed{\text{differentiable}}$ at (a,b)

if $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ with

$\varepsilon_1, \varepsilon_2 \Rightarrow 0$ as $\Delta x, \Delta y \Rightarrow 0$

THM $f(x,y)$ is diffable @ (a,b) if $f_x(a,b), f_y(a,b)$ exist and are cont.

$\boxed{\text{Class}}$: find $L(x,y)$ for xe^{xy} @ $(1,0)$ and approx. $f(1.1, -.1)$.

(1) $f_x = xye^{xy} + e^{xy}$ @ $(1,0) = 1$

$f_y = x^2 e^{xy}$ @ $(1,0) = 1$ $\qquad f(1,0) = 1$

$\Rightarrow L(x,y) = 1 + 1(x-1) + 1(y-0)$

$= x+y.$

(2) $f \cong \underbrace{f(1,0)}_{1} + \underbrace{f_x(1,0)}_{(1)} \cdot \underbrace{\Delta x}_{.1} + \underbrace{f_y(1,0)}_{(1)} \underbrace{\Delta y}_{-.1} \approx \boxed{1}$

Last topic: Differentials. In Calc 1

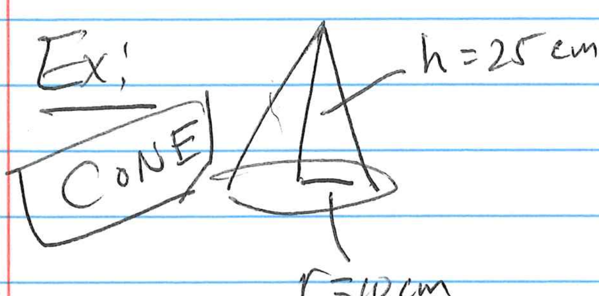
if $y = f(x)$, we wrote $dy = f'(x)\,dx$

think of $dx \approx \Delta x$ indep var

Same in more vars

$\Delta z \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon$ terms

Ex!



possible error of .1 cm in each dir.

Calculate max error in Volume.

Soln

$$V = \frac{\pi r^2 h}{3}$$

$$dV = \frac{\pi}{3}\left[2rh\,dr + r^2\,dh\right]$$

$$\Rightarrow dV \approx \frac{\pi}{3}\left[2 \cdot \overset{r}{10} \cdot \overset{h}{25}\,\overset{dr}{(.1)} + \overset{r^2}{(10)^2}\,\overset{du}{(.1)}\right]$$

$\boxed{\text{Class}}$ Box w/ sides 75 × 60 × 40

w/ .2 max error.

Find max error in dV.

Soln: $V = xyz$

$$dV = xy\,dz + xz\,dy + yz\,dx$$

$$\approx 75 \cdot 60\,(.2) + 75 \cdot 40 \cdot (.2) + 60 \cdot 40 \cdot (.2)$$

$\blacksquare$