## § 14.6

0

**WARM UP:** (implicit diff) 14.5

$F(x,y)=0$ defines $y$ as a function of $x$
(and also defines $x$ as a functn of $y$)

example: $x^2+y^2=6 \Rightarrow x^2+y^2-6=0 \Rightarrow y=\sqrt{6-x^2}$

Chain Rule: $F(x,f(x))=0$

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-F_x}{F_y}$$

Class 27. $y\cos x = x^2+y^2$ Find $dy/dx$

Soln: $x^2+y^2-y\cos x=0$ $\quad F_x = 2x+y\sin x$, $\quad F_y=2y-\cos x$ $\Rightarrow \boxed{\dfrac{2x+y\sin x}{\cos x-2y}}$

23. $w=xy+yz+zx$, $x=r\cos\theta$, $y=r\sin\theta$ $z=r\theta$

Find $\frac{dw}{d\theta}$. Soln: $\frac{dw}{d\theta}=\frac{\partial w}{\partial x}\frac{dx}{d\theta}+\frac{\partial w}{\partial y}\frac{dy}{d\theta}+\frac{dw}{\partial z}\frac{dz}{d\theta}$

Partials
$$\left.\begin{aligned} w_x &= y+z = (r\sin\theta+r\theta)\cdot(-r\sin\theta) &&= dx/d\theta\\ w_y &= x+z = (r\cos\theta+r\theta)\cdot(r\cos\theta) &&= dy/d\theta\\ w_z &= y+\cancel{x} = (r\sin\theta+r\cos\theta)\cdot(r) &&= dz/d\theta\end{aligned}\right.$$

add up to get Soln.

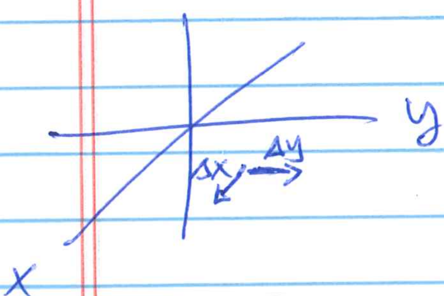
$$= -r^2s^2-\theta r^2 s+r^2c^2+\theta r^2 c+r^2 s+r^2 c$$

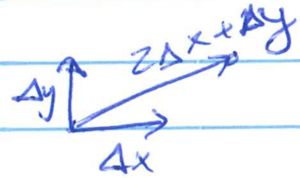
$\left(\begin{aligned} s&=\sin\theta\\ c&=\cos\theta\end{aligned}\right)$

# § 14.6 Directional Derivs, Gradient.

$f_x$ measures change in $f$ if we $\underline{\Delta x}$

$f_y$ " $\Delta y$.




what if we go in directn $2\Delta x + \Delta y$ ?

A: <u>Directional deriv</u>

<u>Def:</u> Let $\vec{u}$ be a unit vector (length = 1) $u = \langle a, b \rangle$

$$\left[ D_u(f)_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h} \right] \quad ★$$

<u>Theorem</u> $D_u(f)(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle \cdot \langle a, b \rangle$.

<u>Pf:</u> $g(h) \stackrel{\Delta}{=} f(x_0+ah, y_0+bh)$. So $g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$ ★

OTOH, by chain rule, $g'(h) = f_x \cdot \underset{a}{\underbrace{\frac{dx}{dh}}} + f_y \underset{b}{\underbrace{\frac{dy}{dh}}}$ ∎

$\boxed{\text{Me}}$ $f(x,y) = x^3 - 3xy + 4y^2$,

$u$ = unit vect in directn $\theta = \pi/6$. Find $D_u(f)(1,2)$

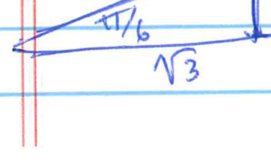
<u>Soln</u> $f_x(1,2) = 3x^2 - 3y @ (1,2) = -3$

$f_y(1,2) = -3x + 8y @ (1,2) = 13$

$\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$\langle -3, 13 \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$= \boxed{\frac{13 - 3\sqrt{3}}{2}}$




$\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$\boxed{\text{Def}}$ The vector $\langle f_x, f_y \rangle = \nabla f$ is the <u>gradient</u>

<u>RMK</u> To maximize $\Delta F$, what do we do?

$$dF = \nabla f \cdot u$$
$$= |\nabla f|\,|u|\,\underbrace{\cos\theta}_{\text{to max, make this } 1}$$

$\cos\theta = 1$ only if $\vec{u}$ has same direction as $\nabla f$!

$\boxed{\text{KEY}}$ "$\nabla f(a,b)$ points in the directn to go in to get max change in $f(x,y)$"

<u>RMK</u> Everything works the same for 3 or more vars.

$\boxed{\text{Class}}$ $f(x,y) = \sin x + e^{xy}$

- If $\vec{u} = \langle a,b \rangle$ find expression for $D_u(f)$
- Find directn to get max $\Delta$ in $f$ at $(2,1)$

<u>Soln:</u> $f_x = \cos x + y e^{xy}$
$f_y = x e^{xy}$ $\Rightarrow D_u(f) = a(\cos x + y e^{xy}) + b\, x e^{xy}$

where $\langle a,b \rangle$ is a unit vector.

At $(x,y) = (2,1)$, $\nabla f = \langle \cos 2 + e^2, e^2 \rangle$

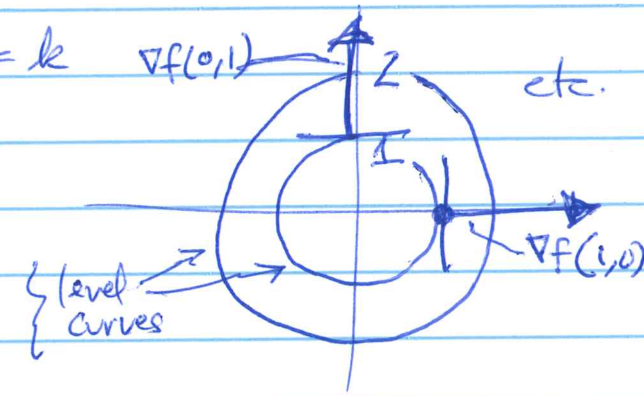
<u>Warning</u> Sometimes people will want directn as a unit vector.

$|\nabla f(a,b)| =$ <u>magnitude</u> of <u>change</u>, when going in directn $\nabla f(a,b)$.

# Tangent planes to level surfaces

Suppose we have $F(x,y,z) = k$
(really think: $w = F(x,y,z)$, so this is the set of $(x,y,z)$ where $F = \underline{\text{const}}\ k$)

<u>In the plane:</u> $x^2+y^2 = k$




$\nabla f = \langle 2x, 2y \rangle$

<u>notice:</u> at $(1,0)$
$\nabla f = \langle 2,0 \rangle$
at $(0,1)$, $\nabla f = \langle 0,2 \rangle$ ! $\nabla f$ is $\perp$ to level curve

Makes sense – if $\nabla f$ points in direction of max $\Delta$, when we're on a level curve (surface) that's where there is <u>NO</u> <u>CHANGE</u>.

<u>Notice</u> $F(x,y,z)=k$. Let $\langle x(t), y(t), z(t) \rangle = r(t)$ be a curve.

$F_x \frac{dx}{dt} + F_y\, dy/dt + F_z \frac{dz}{dt} = 0 = \nabla f \cdot \vec{r}\,'(t)$

$\Rightarrow$ if $r(t)$ is a curve on the surface, $\nabla f \perp$ to <u>velocity</u>


$\Rightarrow \nabla f$ is $\perp$ to <u>surface</u>

this makes sense – if $z - f(x,y) = 0$,
$\nabla f = \langle -f_x, -f_y, 1 \rangle = \underline{\underline{\text{normal}}}$ !

$\boxed{\text{Class}}$

Find eqn of tangent plane and normal line to $yz = \ln(x+z)$ @ $(0,0,1)$

$yz - \ln(x+z) = 0$

Soln: $\nabla(f) = \left\langle -\frac{1}{x+z},\ z,\ y - \frac{1}{x+z} \right\rangle$ at $(0,0,1)$

$= \langle -1, 1, -1 \rangle$

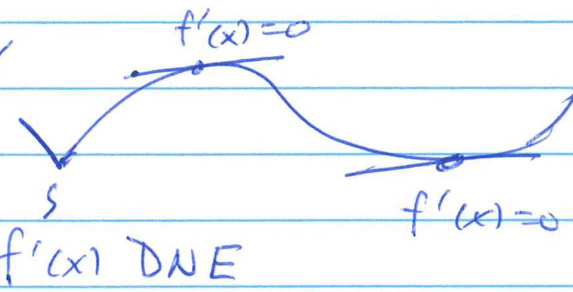
Normal line: $(0,0,1) + t(-1,1,-1)$

Tan plane:

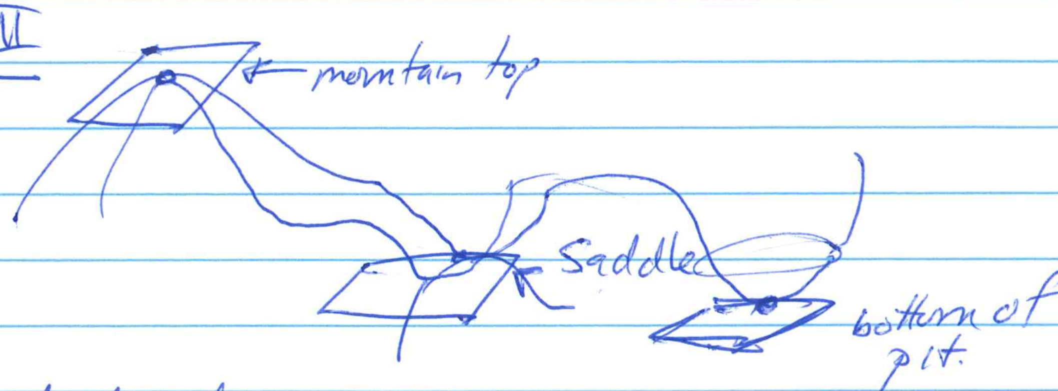
$(x-0, y-0, z-1) \cdot (-1,1,-1) = 0$

Look ahead min/max

Calc 1 $f'(x) = 0$ or DNE




Calc III




instead of $f'(x)=0$, we need

$f_x, f_y$ both zero or DNE

But thats not enough ....