# 14.7

## WARM UP.

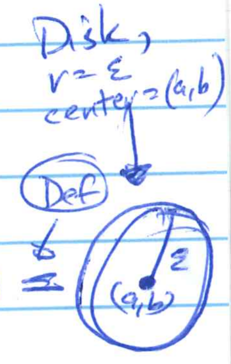
Find max rate of change + direction (unit vector) for (24) $f(x,y,z) = x\ln(yz)$ @ $(1, 2, \frac{1}{2})$.

Soln: $\nabla f = \langle \ln yz, xz/yz, xy/yz \rangle$ @ $(1, 2, \frac{1}{2})$

$= \langle \ln(1), \ln(\frac{1}{2}), \ln(2) \rangle = \langle 0, -\ln(2), \ln(2) \rangle$

$|\nabla f| = \sqrt{2\ln(2)^2} = \sqrt{2}\ln(2)$, dir $= \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ ∎

Last time, we did a look ahead.

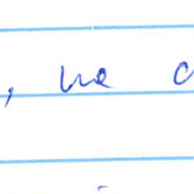



**Def:** (a,b) is a local min (max) if $f(a,b) \le (\ge) f(x,y)\ \forall (x,y) \in D_\varepsilon(a,b)$

**Thm** If (a,b) is a local min/max, ~~[illegible]~~ and partials exist @ (a,b) then $f_x(a,b) = 0 = f_y(a,b)$

**Pf:** slice, use calc 1


Warning

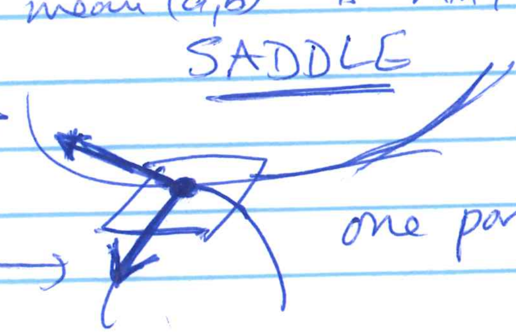


$f_x(a,b) = 0 = f_y(a,b)$ does **Not** mean (a,b) is min/max.

**SADDLE**

one partial + $f_y$ +

$f_x$ − →



one partial −

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**Def:** p is a <u>critical pt</u> if both partials are zero, or if one does not exist

**Example** (me) find crit pts of $x^2+y^2-2x-6y+14$

$f_x = 2x-2 = 0 \Leftrightarrow x=1$

$f_y = 2y-6 = 0 \Leftrightarrow y=3$

<u>crit pt</u> is $(1,3)$

$\boxed{\text{Class:}}$ find all crit pts of

$f(x,y) = x^4+y^4-4xy+1$

$f_x = 4x^3-4y = 0 \Rightarrow \left\{ \begin{array}{l} y=x^3 \\ x=y^3 \end{array} \right\}$ <u>combine</u>

$f_y = 4y^3-4x = 0 \Rightarrow$ $y^3=x^9$

$x-x^9=0$

$x(1-x^8)=0 \quad x=0$

or $x^8=1 \Rightarrow x=\pm 1$

| x | y |
|---|---|
| 0 | 0 |
| 1 | 1 |
| -1 | -1 |

3 <u>crit pts.</u>

~~BUT NOT~~

How tell min/max? <u>Hessian</u> $\boxed{\text{THM}}$

$$\det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} > 0 \ \boxed{\text{AND}} \ f_{xx} \text{ or } f_{yy} > 0 \text{ min}$$

$$\text{"} \quad < 0 \text{ max}$$

$< 0$ <u>Saddle</u>

$= 0$ <u>inconclusive</u>

(if Hessian +, $f_{xx}$ +)

**Proof:** Let $u = hi + kj$

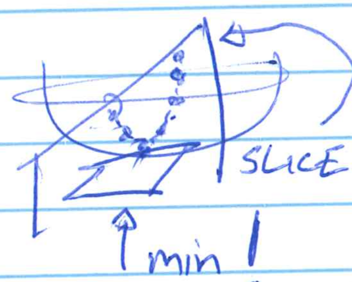
$$D_u(f) = h f_x + k f_y$$

do twice (not squaring)

$$D_u^2 = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$$

$$= \underbrace{f_{xx}}_{+}\underbrace{\left(h + \frac{f_{xy}}{f_{xx}}\right)^2}_{+} + \underbrace{\frac{k^2}{f_{xx}}}_{+}\Big(\underbrace{f_{xx} f_{yy} - f_{xy}^2}_{+ \text{ by assumption}}\Big)$$

$\Rightarrow D_u^2$ always +, any slice




$\boxed{\text{Me}}$ For Example 1, min/max/saddle?

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = 2$

$$\det \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 = + \quad \text{POSITIVE its min or max}$$

$f_{xx}$ + $\Longrightarrow$ min!

$\boxed{\text{CLASS}}$ For Example 2 $\quad f_{xx} = 12x^2, \; f_{xy} = -4, \; f_{yy} = 12y^2$

$$\det \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y^2 - 16 \qquad \left\{ \begin{matrix} 0 & 0 \\ 1 & 1 \\ -1 & -1 \end{matrix} \right\}$$

$-$ at $(0,0)$ $\leftarrow$ SADDLE

$+$ at $(1,1), (-1,-1)$ $\leftarrow$ POSITIVE = min or max

$f_{xx} = 12x^2$ + $\Rightarrow$ both min

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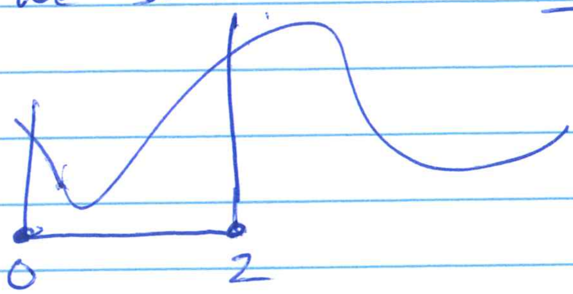
Those were local min/max. In calc 1 we saw on a closed interval



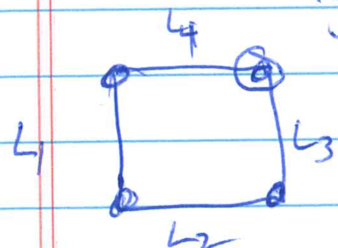

we had to check partials + endpoints

**Theorem**: If $D$ is a closed set in $\mathbb{R}^2$ then if $f(x,y)$ is continuous on $D$, it assumes its max/and min either at a critical pt, or on boundary.

Example (me) Find min/max of $x^2-2xy+2y$ on the closed unit square.

Soln: $f_x = 2x-2y = 0 \Rightarrow y = x \Rightarrow y = 1$
$f_y = 2-2x = 0 \Rightarrow x = 1$

crit pts $(1,1)$




also have to check

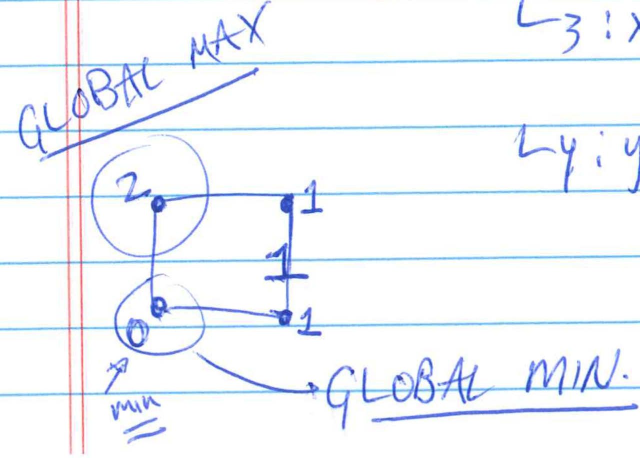
$L_1: x=0, y=0..1 \Rightarrow f(0,y) = 2y$ — $\boxed{\text{CALC 1}}$ min @ $(0,0)$, max @ $(0,1)$

$L_2: y=0, x=0..1 \Rightarrow f(x,0) = x^2$ — min @ $(0,0)$, max @ $(1,0)$

$L_3: x=1, y=0..1 \Rightarrow f(1,y) = \underline{1}$ CONST!

$L_4: y=1, x=0..1 \Rightarrow f(x,1) = x^2-2x+2$ — min @ $(1,1)$, max @ $(0,1)$

$\underline{2x-2 \quad x=1}$ min



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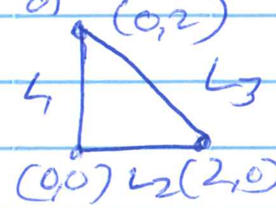
**Class** 31 Find ABS min/max of $x^2+y^2-2x$ on closed triangle:




1) $f_x = 2x - 2 \stackrel{\text{set}}{=} 0 \Rightarrow x = 1$
$f_y = 2y \Rightarrow 0 \Rightarrow y = 0$ crit pt = (1,0)

$f_{xx}=2,\ f_{yy}=2,\ f_{xy}=0$ $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$ +, $f_{xx}$ +

local min @ (1,0)

$\boxed{f(1,0) = -1}$

$L_1$: $x=0,\ y=0..2 \Rightarrow f(0,y)=y^2$, $f'=2y=0$ @ $y=0$

$f(0,0) = 0$
$f(0,2) = 4$

{ deriv + endpts get checked $\Rightarrow$ calc 1

$L_2$: $y=0,\ x=0..2 \Rightarrow f(x,0)=x^2-2x$, $f'=2x-2=0$ @ $x=1$

$f(0,0)$ ✓, $f(2,0)=0$, $f(1,0)=-1$

$L_3$: $x+y=2 \Rightarrow y=2-x \Rightarrow f = x^2+(2-x)^2-2x$
$= x^2+4-4x+x^2-2x$
$= 2x^2-6x+4$

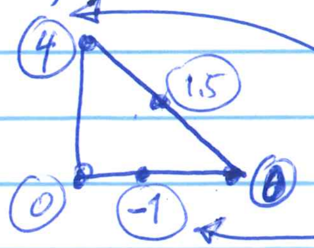
already checked endpts (2,0), (0,2)

$f' = 4x-6 = 0 \Rightarrow x = 3/2,\ y = 3/2$

$f(3/2, 3/2) = (3/2)^2 + (3/2)^2 - 2(3/2)$
$= 9/4 + 9/4 - 3$

$\frac{18}{4} - \frac{12}{4} = \frac{6}{4} = \boxed{1.5}$

Value of f:




ABS MAX.

MIN.