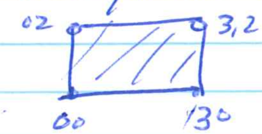
# 14.8

WARM UP: Find critical pts (identify type) and absolute min / max of

$f(x,y) = x^2 - 2xy + 2y$ on




<u>SOLN</u>

$f_x = 2x - 2y$ } $= 0 \Rightarrow x = y$

$f_y = -2x + 2$ } $= 0 \Rightarrow x = 1$

$f_{xx} = 2$ Crit pt @ $(1,1)$

$f_{xy} = -2$ $H = \begin{vmatrix} 2 & -2 \\ -2 & 0 \end{vmatrix} = -4$, so $(1,1)$ is a saddle //

$f_{yy} = 0$

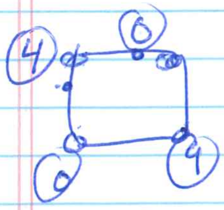
Now check edges

$y = 0 \Rightarrow f = x^2$ max @ $(3,0)$ $(= 9)$, min at $(0,0) = \underline{0}$

$y = 2 \Rightarrow f = x^2 - 4x + 4$ max @ $(0,2) (= 4)$, min @ $(2,2) = \underline{0}$

$x = 0 \Rightarrow f = 2y$ max at $(0,2)$, min at $(0,0) = 0$

$x = 3 \Rightarrow f = 9 - 4y$ max @ $(3,0) = 9$, min @ $(3,2) = 1$




<u>Soln</u> mins at $(0,0)$, $(2,2)$ [both have $f = 0$]

max at $(3,0)$ [has $f = 9$]

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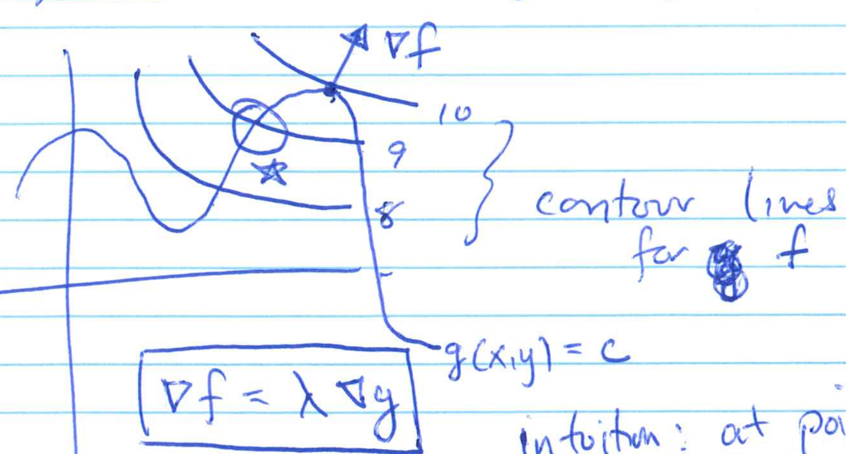
## § 14.8 Constrained Min / Max [LaGrange]

Maximize $f(x,y,z)$ subject to constraint $g(x,y,z) = c$.

<u>Easier picture</u> max $f(x,y)$ subject to $g(x,y) = c$.




$\boxed{\nabla f = \lambda \nabla g}$

intuition: at point like ☆, can stay on $g(x,y) = c$ and change value of $f$.

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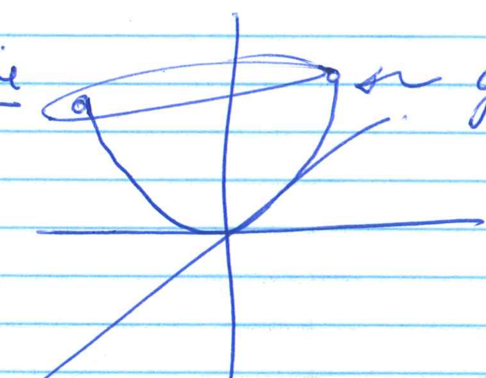
So, the key insight is that at a point where $f$ is max/min subject to $g = \text{const}$

the gradients of $f, g$ are $\parallel$ (= const. multiples)

**Example 2 (3D)**

- max $f(x,y,z) = -z$
- subject to $g(x,y,z) = z - x^2 - y^2 = 0$

picture



$g=0$ = parabolic cereal bowl

clearly $(-z)$ is max at $(x,y) = (0,0)$.

Lets check

$$\nabla f = \langle 0, 0, -1 \rangle$$
$$\nabla g = \langle -2x, -2y, 1 \rangle$$

$\nabla f = \lambda \nabla g$

$$\underbrace{\langle 0,0,-1 \rangle = \lambda \langle -2x, -2y, 1 \rangle}_{\text{system of eqns}}$$

$$\begin{cases} -2x\lambda = 0 \\ -2y\lambda = 0 \\ -1 = \lambda \end{cases}$$

$\lambda = -1 \Rightarrow x = y = 0$

which is indeed our solution!

AND $z - x^2 - y^2 = 0$.

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$\boxed{\text{Class}}$

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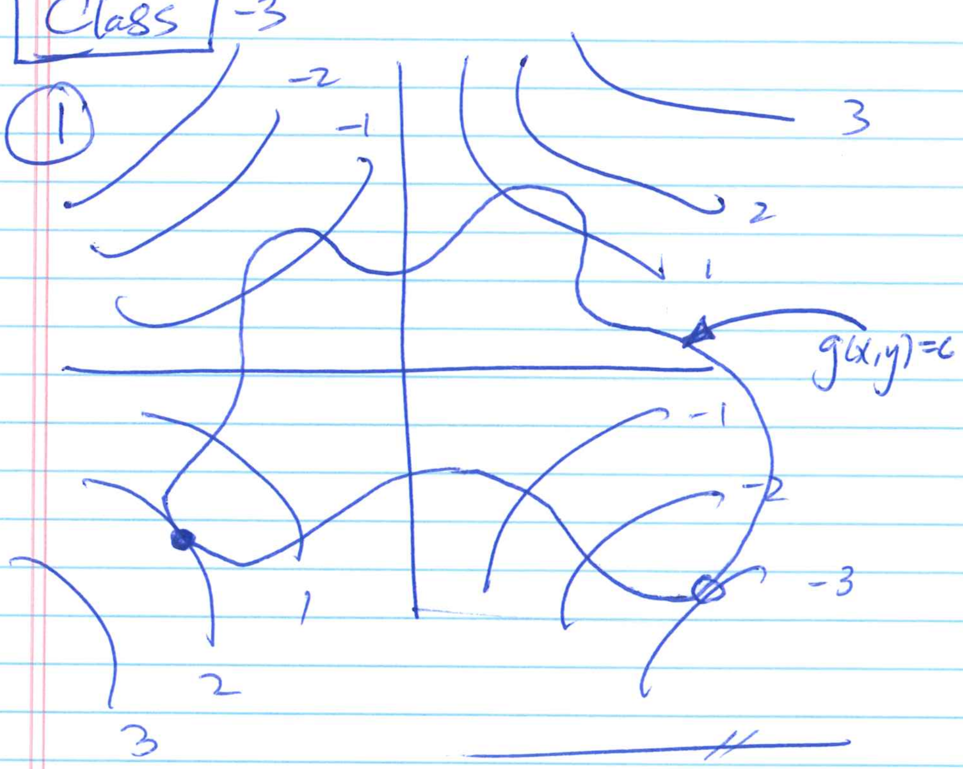



Find max/min on $g(x,y)=c$ given contour lines for $f(x,y)$

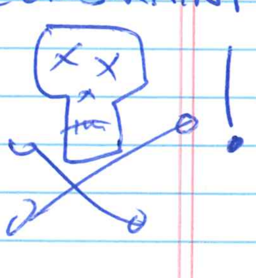
Soln: max at •
min at O

2 max / min of $x^2+2y^2 = f(x,y)$ subject to $x^2+y^2=1 = g(x,y)$

$\nabla f = \langle 2x, 4y \rangle \qquad \nabla g = \langle 2x, 2y \rangle$

$$\nabla f = \lambda \nabla g \Rightarrow \left.\begin{array}{l} 2x = \lambda 2x \longrightarrow \lambda = 1 \text{ or } x = 0 \\ 4y = \lambda 2y \longrightarrow y = 0 \end{array}\right\}$$

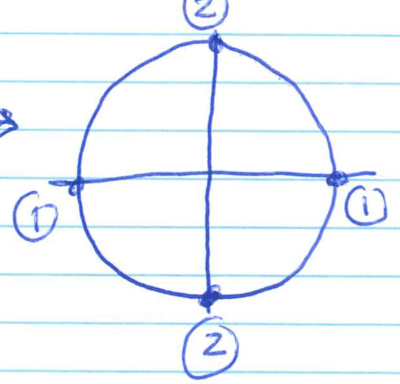
ALWAYS $\Rightarrow$ REMEMBER CONSTRAINT



!

AND $x^2+y^2=1$

Solutions are where either $y=0$ or $x=0$




Value of f is in circle

(PLUG IN POINTS TO f)

So mins are at $(-1,0), (1,0)$

max at $(0,2), (0,-2)$

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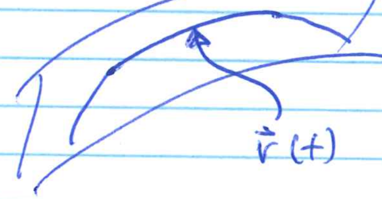
**Proof of LaGrange:** Let $S = \{g(x,y,z) = c\}$ be a surface, $\vec{r}(t) = (x(t), y(t), z(t))$ a curve on $S$.




$g(\vec{r}(t)) = c$

$\vec{r}'(t)$

CHAIN RULE $\Rightarrow \langle g_x, g_y, g_z \rangle \cdot \langle x', y', z' \rangle = 0$

$\Rightarrow \nabla g$ is $\perp$ to Tangent plane to $S$.

OTOH, if $f(\vec{r}(t))$ is a max, then $\frac{df}{dt} = 0$, so (CHAIN AGAIN!) $\nabla f \cdot \vec{r}'(t) = 0$ $\Rightarrow \nabla f$ and $\nabla g$ point in same directn.

Notice: this is for all $r(t)$ on surface, so the $r'(t)$ span the tangent plane

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This works for more constraints: max $f(x,y,z)$ subject to $g_1(x,y,z) = c_1$, $g_2(x,y,z) = c_2$.

Example: max $x^2 + y^2 + z^2$ subject to $y = 0$, $z = 0$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 0,1,0 \rangle + \mu \langle 0,0,1 \rangle$$

$$\left.\begin{array}{l} 2x = 0 \\ 2y = \lambda \\ 2z = \mu \end{array}\right\} \Longrightarrow$$

BUT ON OUR CONSTRAINTS, $y = z = 0$

$\Longrightarrow \left\{ f \text{ is just } x^2 = 0, \text{ min at } x = 0, \text{ no max.} \right\}$

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Better example of multiple onstraints (Ex 5, book)

Max $x+2y+3z$ subject to

$$g_1 = x-y+z=1$$
$$g_2 = x^2+y^2=1$$

$$\nabla f = \langle 1,2,3\rangle = \lambda \nabla g_1 + \mu \nabla g_2$$
$$= \lambda\langle 1,-1,1\rangle + \mu\langle 2x, 2y, 0\rangle$$

$$\Rightarrow \left.\begin{aligned} 1 &= \lambda + 2x\mu \\ 2 &= -\lambda + 2y\mu \\ 3 &= \lambda + 0\mu \end{aligned}\right\} \quad \lambda = 3 \rightarrow \quad \left.\begin{aligned} 2x\mu &= -2 \\ 2y\mu &= 5 \end{aligned}\right\} \begin{aligned} x &= -1/\mu \\ y &= 5/2\mu \end{aligned}$$

PLUG IN

$$x^2+y^2=1$$
$$\left(-\frac{1}{\mu}\right)^2 + \left(\frac{5}{2\mu}\right)^2 = 1$$
$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$
$$\frac{4+25}{4\mu^2} = 1$$
$$\frac{29}{4} = \mu^2$$
$$\boxed{\pm\frac{\sqrt{29}}{2}} = \mu$$

$$x = \pm\frac{2}{\sqrt{29}}$$

$$y = \pm\frac{5}{\sqrt{29}}$$

$$z = 1 \pm 7/\sqrt{29} \quad \left(\begin{matrix}\text{use}\\ g_2\end{matrix}\right)$$

NOW PLUG THESE IN TO $f$, and check.

$\boxed{\text{POINT}}$ → La Grange gives us a finite set of points that are **Candidates**, we plug in and check.